

DENSITY ESTIMATES FOR VECTOR MINIMIZERS AND APPLICATIONS

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Abstract

We extend the Caffarelli-Cordoba estimates to the vector case in two ways, one of which has no scalar counterpart, and we give a few applications for minimal solutions.

1 Introduction

This paper is concerned with solutions to the system

$$(1.1) \quad \Delta u - W_u(u) = 0, \quad u : D \rightarrow \mathbb{R}^m$$

$D \subset \mathbb{R}^n$, where $D = \mathbb{R}^n$ is an important special case, $W : \mathbb{R}^m \rightarrow \mathbb{R}$, $W \geq 0$, with regularity specified later, and $W_u = (\frac{\partial W}{\partial u_1}, \dots, \frac{\partial W}{\partial u_m})^\top$.

Unlike the scalar case $m = 1$, where for a class of results, the form of the *potential* W is immaterial, for the system the connectedness of $\{W = 0\} \neq \emptyset$ plays a major role. Distinguished examples are: (a) the phase transition model or vector Allen-Chan equation, where W has a finite number N of global minima a_1, \dots, a_N (Baldo [10], Bronsard and Reitich [12]), (b) the Ginzburg-Landau system $\Delta u - (|u|^2 - 1)u = 0$ (Bethuel, Brezis and Helein [11]) and (c) the phase separation system $\Delta u - \sum_{j \neq i} u_i u_j = 0$ (Caffarelli and Lin [15]) and its variants.

System 1.1 is the Euler-Lagrange equation for the functional

$$(1.2) \quad J_D(u) = \int_D \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx.$$

In the present paper we limit ourselves to uniformly bounded solutions to (1.1) that are *minimal* in the sense that

$$(1.3) \quad J_\Omega(u) = \min_v J_\Omega(v), \quad v = u \text{ on } \partial\Omega$$

for every Ω open, bounded Lipschitz $\Omega \subset D$.

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The basic estimate for such solutions is

$$(1.4) \quad \int_{B_B(x_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \leq CR^{n-1},$$

$B_R(x_0)$ the R -ball in \mathbb{R}^n , center x_0 , $B_R(x_0) \subset D$.

The key hypothesis in our theorems is

$$(1.5) \quad W(a) = 0, \quad a \text{ isolated in } \{W = 0\}$$

This assumption excludes examples (b) and (c) above. It is well known that the phase transition model is linked to minimal surfaces ($m = 1$) and Plateau Complexes ($m \geq 2$). In particular in the vector case entire solutions to (1.1) are linked to singular minimal cones which unlike planes have additional hierarchical structure (Alikakos [3]).

The main purpose of this paper is the various extensions of the Caffarelli-Cordoba density estimates [14] to the vector case. In the scalar case, among other things, these estimates refine the linking of the phase transition model to minimal surfaces and have played a major role in the resolution of De Giorgi conjecture in higher dimensions (Savin [23]). Other extensions to the density estimates in different contexts have been provided by Farina and Valdinoci [18], Savin and Valdinoci [24],[25], and Sire and Valdinoci [26]. Set

$$(1.6) \quad \begin{cases} A_R = \int_{B_R \cap \{|u-a| \leq \lambda\}} W(u) dx, \\ V_R = \mathcal{L}^n(B_R \cap \{|u-a| > \lambda\}) \end{cases}$$

where \mathcal{L}^n stands for the n -dimensional Lebesgue measure. Note that A_R satisfies $A_R \leq CR^{n-1}$ by (1.4). In the context of diffuse interfaces A_R measures interface area while V_R enclosed volume ([14]).

Theorem A. *Under (1.5) and regularity of W as in (HA) in the next section, for $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$, minimal, $\|u\|_{L^\infty} < \infty$, the following holds for $0 < \lambda < \text{dist}(a, \{W = 0\} \setminus \{a\})$:*

$$\text{If} \quad V_1 \geq \mu_0 > 0,$$

$$\text{then} \quad V_R \geq CR^n, \quad R \geq 1, \quad C = C(\mu_0, \lambda, \|u\|_{L^\infty}).$$

The new points in the proof of Theorem A are the *polar form*

$$(1.7) \quad \begin{aligned} u(x) &= a + q^u(x) \nu^u(x), \\ q^u(x) &= |u(x) - a|, \quad \nu^u(x) = \frac{u(x) - a}{|u(x) - a|}, \end{aligned}$$

the choice of the test functions which are limited to perturbations of the modulus q^u and keep ν^u fixed,

$$(1.8) \quad \sigma = a + q^\sigma \nu^u, \quad q^\sigma = \min\{q^h, q^u\},$$

and the resulting identity

$$(1.9) \quad \begin{aligned} & \frac{1}{2} \int_{B_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dx \\ &= J_{B_R}(u) - J_{B_R}(\sigma) + \frac{1}{2} \int_{B_R} ((q^\sigma)^2 - (q^u)^2) |\nabla \nu^u|^2 dx + \frac{1}{2} \int_{B_R} (W(\sigma) - W(u)) dx \\ &\leq \frac{1}{2} \int_{B_R} (W(\sigma) - W(u)) dx \end{aligned}$$

where minimality on balls was used in the last inequality. The proof of Theorem A otherwise follows closely the argument in Caffarelli-Cordoba [14].

We give a number of applications of Theorem A. We mention here a few and refer the reader to the main body of the paper for the precise statements.

(i) Lower Bound

For the phase transition model (a) above, under the hypotheses of Theorem A, and provided u is not a constant, the lower bound holds

$$(1.10) \quad \int_{B_R(x_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \geq CR^{n-1}, \quad R \geq R(x_0),$$

$C > 0$ independent of x_0 .

We recall that for all nonconstant solutions to (1.1) and any $W \geq 0$ which allows $u \in W_{\text{loc}}^{1,2} \cap L^\infty$ the estimate

$$(1.11) \quad \int_{B_R(x_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \geq CR^{n-2}$$

holds, and that (1.11) can not in general be improved (Alikakos [2]). In light of (1.4) estimate (1.10) is optimal.

(ii) Liouville-Rigidity Theorem

If $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a bounded solution to (1.1), minimal, and if either $\{W = 0\} = \{a\}$, or $\inf_x d(u(x), \{W = 0\} \setminus \{a\}) > 0$, Then

$$u \equiv a$$

This was proved in Fusco [20] with a different, though related method.

(iii) Linking

For global minimizers of $J_\epsilon(u) = \int_D \left(\frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \right) dx$, for D open, bounded, with Dirichlet conditions on ∂D , and W with exactly two minima, $W(a_1) = W(a_2) = 0$, $W > 0$ on $\mathbb{R}^m \setminus \{a_1, a_2\}$, $S_\epsilon = \{|u_\epsilon - a_j| = \gamma\}$, $\gamma \in (0, |a_1 - a_2|)$ converges uniformly as $\epsilon \rightarrow 0^+$ to the minimal partition with Dirichlet conditions.

The proof is completely analogous to the corresponding scalar result in Caffarelli-Cordoba [14].

Entire equivariant (minimal) solutions to (1.1) correspond to minimal cones and possess a hierarchical structure at least for a class of symmetries. They were established by Bronsard, Gui and Schatzman [13] for triple junctions, $n = m = 2$, and by Gui and Schatzman [22] for quadruple junctions ($n = m = 3$) and for general n, m in a series of papers [5] [4] [19]. In the papers [13] [22] the hierarchical structure is built in, while in [5] [4] [19] can be deduced a posteriori (see [8]).

Our next theorem concerns an aspect that has no scalar counterpart. We look at the simplest possible set up for this kind of result. Consider (1.1) in the class of symmetric solutions

$$u(\hat{x}) = \hat{u}(x)$$

where for $z \in \mathbb{R}^d$ we denote by \hat{z} the reflection of z in the plane $\{z_1 = 0\}$,

$$\hat{z} = (-z_1, z_2, \dots, z_d),$$

and we take W a C^3 potential, symmetric $W(u) = W(\hat{u})$, $u \in \mathbb{R}^m$, and with exactly two minima $W(a_-) = W(a_+) = 0$, $W > 0$ on $\mathbb{R}^m \setminus \{a_+, a_-\}$. Under hypotheses of nondegeneracy for a_+, a_- there is such a symmetric solution, minimal in the symmetric class, and satisfying the estimate

$$|u - a_+| + |\nabla u| \leq K e^{-kx_1}, \quad x_1 \geq 0.$$

Consider the *Action*

$$A(v) = \int_{\mathbb{R}} \left(\frac{1}{2} |v_s|^2 + W(v) \right) ds$$

for symmetric $v \in W_{loc}^{1,2}(\mathbb{R}; \mathbb{R}^m) \cap L^\infty(\mathbb{R}; \mathbb{R}^m)$, that connect at infinity the minima, $\lim_{s \rightarrow \pm\infty} v(s) = a_\pm$.

The key hypotheses in our theorems is that A has a hyperbolic global minimum e in the symmetric class. Following [8] we define the *Effective-Potential*

$$(1.12) \quad \mathcal{W}(v(\cdot)) = A(v(\cdot)) - A(e(\cdot)) \geq 0$$

and thus we have that

$$(1.13) \quad \mathcal{W}(e) = 0, \quad e \text{ isolated in } \{\mathcal{W} = 0\}$$

(cfr. (1.5)) above.

The basic estimate in the present context is

$$(1.14) \quad 0 \leq \int_{C_R(y_0)} \left(\left(\frac{1}{2} |\nabla u|^2 + W(u) \right) - A(e) \right) dx \leq C R^{n-2},$$

$C_R(y_0)$ the cylinder $\mathbb{R} \times \mathcal{B}_R(y_0)$, $\mathcal{B}_R(y_0)$ the R -ball in \mathbb{R}^{n-1} with center at $y_0 \in \mathbb{R}^{n-1}$, $x = (s, y)$. Set

$$\|f\| = \left(\int_{\mathbb{R}} |f(s)|^2 ds \right)^{\frac{1}{2}}, \quad f : \mathbb{R} \rightarrow \mathbb{R}^m$$

and by analogy to (1.6)

$$(1.15) \quad \begin{cases} \mathcal{A}_R = \int_{\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e(\cdot)\| \leq \lambda\}} \mathcal{W}(u) dy, \\ \mathcal{V}_R = \mathcal{L}^{n-1}(\mathcal{B}_R \cap \{y : \|u(\cdot, y) - e(\cdot)\| > \lambda\}). \end{cases}$$

Note that $\mathcal{A}_R \leq C R^{n-2}$ by (1.14).

Theorem B. *Let u symmetric, and minimal in the symmetry class, as above. Under (1.13) in the $\|\cdot\|$ sense, there is $\lambda^* > 0$ such that, for $0 < \lambda < \lambda^*$ the following holds:*

$$\begin{aligned} \text{If} \quad & \mathcal{V}_1 \geq \mu_0 > 0, \\ \text{then} \quad & \mathcal{V}_R \geq C R^{n-1}, \quad R \geq 1, \quad C = C(\mu_0, \lambda, \|u\|_{L^\infty}). \end{aligned}$$

The proof of Theorem B, following [8], implements the *polar form*

$$\begin{aligned} u(\cdot, y) &= e(\cdot) + q^u(y)\nu^u(\cdot, y), \\ q^u(y) &= \|u(\cdot, y) - e(\cdot)\|; \quad \nu^u(\cdot, y) = \frac{u(\cdot, y) - e(\cdot)}{\|u(\cdot, y) - e(\cdot)\|} \end{aligned}$$

and utilizes test functions that vary only q^u ,

$$\sigma(\cdot, y) = e(\cdot) + q^\sigma(y)\nu^u(\cdot, y), \quad q^\sigma = \min\{q^h, q^u\}$$

and employs the identity

$$\begin{aligned} (1.16) \quad & \frac{1}{2} \int_{\mathcal{B}_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dy \\ &= J_{C_R}(u) - J_{C_R}(\sigma) + \frac{1}{2} \int_{\mathcal{B}_R} \left((q^\sigma)^2 - (q^u)^2 \right) \sum_{i=1}^{n-1} \left\| \frac{\partial \nu^u}{\partial y_i} \right\|^2 dy + \int_{\mathcal{B}_R} (\mathcal{W}(\sigma) - \mathcal{W}(u)) dy \\ &\leq \int_{\mathcal{B}_R} (\mathcal{W}(\sigma) - \mathcal{W}(u)) dy. \end{aligned}$$

where in the last inequality minimality with respect to cylinders was used. Thus the proof, *mutatis mutandis*, follows Caffarelli-Cordoba [14].

We now mention some of the applications of Theorem B and refer the reader to the main body of the paper for more information and precise statements.

- (i) Assume that the Action A has exactly two global minima e_-, e_+ , $\mathcal{W}(e_-) = \mathcal{W}(e_+) = 0$, $\mathcal{W} > 0$ otherwise, where e_-, e_+ satisfy the hypotheses of e above. Assume for u the hypotheses of Theorem B. Then for $0 < \theta < \|e_- - e_+\|$ the following is true:

If

$$(1.17) \quad \mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_-(\cdot)\| \leq \theta\}) \geq \mu_0 > 0$$

Then

$$(1.18) \quad \mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e_-(\cdot)\| \leq \theta\}) \geq CR^{n-1}$$

for $R \geq 1$, $C = C(\mu_0, \theta, \|u\|_{L^\infty})$, with a similar statement for e_+ .

- (ii) Assume the hypothesis of Theorem B and suppose that either $\{\mathcal{W} = 0\} = \{e\}$ or $\inf_y \|u(\cdot, y) - (\{\mathcal{W} = 0\} \setminus \{e\})\| > 0$, then

$$u \equiv e.$$

This was proved in [8] under the hypothesis $\{\mathcal{W} = 0\} = \{e\}$ with a different though related method.

We recall that Alama, Bronsard and Gui in [1] have established, under the hypothesis of (i) above, the existence of a solution $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ converging to a_\pm as $s \rightarrow \pm\infty$, and converging to e_\pm as $y \rightarrow \pm\infty$. Thus there are solutions genuinely higher dimensional connecting e_+ and e_- . The paper is structured as follows. In Part I Theorem A is stated and proved and its applications are presented in individual sections. Similarly in Part II Theorem B is stated and proved, followed by its applications.

PART I

2 Theorem A

2.1 Hypotheses and Statement

(HA) The potential $W : \mathbb{R}^m \rightarrow \mathbb{R}$ is nonnegative and $W(a) = 0$ for some $a \in \mathbb{R}^m$. Moreover $W \in C^\alpha(\mathbb{R}^m; \mathbb{R}) \cap C^1(\mathbb{R}^m \setminus \{a\}; \mathbb{R})$.

If $0 < \alpha < 2$ we assume

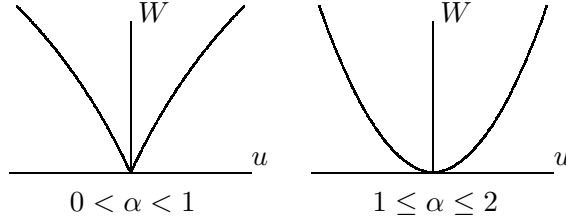
$$(2.1) \quad W_u(a + \rho\nu) \cdot \nu \geq C^* \rho^{\alpha-1}, \quad \text{for } 0 < \rho \leq \rho_0, |\nu| = 1$$

where \cdot denotes the Euclidean inner product in \mathbb{R}^m , C^* a positive constant.

If $\alpha = 2$ we assume, for some constant $C_0 > 0$,

$$(2.2) \quad W_{uu}(a)\nu \cdot \nu \geq C_0 > 0, \quad \text{for } |\nu| = 1.$$

The figure below shows the behavior of W for different values of α .



(HB) $u : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $u \in W_{\text{loc}}^{1,2}(D; \mathbb{R}^m) \cap L^\infty(D; \mathbb{R}^m)$, is *minimal* in the sense that

$$(2.3) \quad J_\Omega(u) \leq J_\Omega(u + v), \quad \text{for } v \in W_0^{1,2}(\Omega; \mathbb{R}^m)$$

for every open bounded set $\Omega \subset D$, where

$$(2.4) \quad J_\Omega(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx.$$

$$(2.5) \quad |u - a| < M, \quad |\nabla u| < M, \quad \text{on } \mathbb{R}^n.$$

Note: In the proof of Theorem A we utilize minimality only on balls.

For each $z \in \mathbb{R}^k$, $k \geq 1$ and $r > 0$ we let $B_r(z) \subset \mathbb{R}^k$ be the open ball of center z and radius r and B_r the ball centered at the origin. We denote by $\mathcal{L}^k(E)$ the k -dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^k$.

Theorem A. Under hypothesis (HA) and (HB) above, for any $\mu_0 > 0$ and any $0 < \lambda < d_0 = \text{dist}(a, \{W = 0\} \setminus \{a\})$, the condition

$$(2.6) \quad \mathcal{L}^n(B_1(x_0) \cap \{|u - a| > \lambda\}) \geq \mu_0, \quad ,$$

provided $B_R(x_0) \subset \Omega$, implies the estimate

$$(2.7) \quad \mathcal{L}^n(B_R(x_0) \cap \{|u - a| > \lambda\}) \geq CR^n, \quad \text{for } R \geq 1$$

where $C = C(\mu_0, \lambda, M)$, C independent of x_0 and independent of u .

As in [14] Theorem A has the following important consequence

Theorem 2.1. *Assume there are $a_1 \neq a_2 \in \mathbb{R}^m$ such that*

$$W(a_1) = W(a_2) = 0, \quad W(u) > 0, \quad \text{for } u \notin \{a_1, a_2\}$$

and assume that (HA) holds at $a = a_j$, $j = 1, 2$. Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a minimizer in the sense of (HB). Then, given $0 < \theta < |a_1 - a_2|$ the condition

$$(2.8) \quad \mathcal{L}^n(B_1(x_0) \cap \{|u - a_1| \leq \theta\}) \geq \mu_0 > 0$$

implies the estimate

$$(2.9) \quad \mathcal{L}^n(B_R(x_0) \cap \{|u - a_1| \leq \theta\}) \geq CR^n, \quad \text{for } R \geq 1$$

where $C > 0$ depends only on μ_0 , θ and M . An analogous statement applies to a_2 .

Proof. Since $|u - a_1| \leq \theta$ implies $|u - a_2| > |a_1 - a_2| - \theta = \lambda > 0$ the assumption (2.8) implies

$$\mathcal{L}^n(B_1(x_0) \cap \{|u - a_2| > \lambda\}) \geq \mu_0.$$

Therefore Theorem A yields

$$\mathcal{L}^n(B_R(x_0) \cap \{|u - a_2| > \lambda\}) \geq CR^n, \quad \text{for } R \geq 1.$$

To conclude the proof we observe that

$$\{|u - a_2| > \lambda\} = \{|u - a_1| \leq \theta\} \cup (\{|u - a_1| > \theta\} \cap \{|u - a_2| > \lambda\})$$

and therefore $W(u) > 0$, for $u \notin \{a_1, a_2\}$ and Lemma 2.2 below imply $\mathcal{L}^n(B_R(x_0) \cap (\{|u - a_1| > \theta\} \cap \{|u - a_2| > \lambda\})) \leq CR^{n-1}$. \square

Note: We note that the argument above when applied to potentials W that vanish at more than two points: $W(a_1) = \dots = W(a_N) = 0$, $N \geq 3$, provides estimates (2.9) only for two of the minima, even if (2.8) holds for all N of them. The selection of the particular two minima depends in general on R .

2.2 The Proof of Theorem A

1. The Polar Form

We will utilize the *polar form* of a vector map $u \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, $A \subset \mathbb{R}^n$ open and bounded,

$$(2.10) \quad u(x) = a + q^u(x)\nu^u(x)$$

where

$$(2.11) \quad q^u(x) = |u(x) - a|, \quad \nu^u(x) = \begin{cases} \frac{u(x) - a}{|u(x) - a|}, & \text{if } u(x) \neq a, \\ 0, & \text{if } u(x) = a. \end{cases}$$

We have [9] $q^u \in W^{1,2}(A) \cap L^\infty(A)$ and $\nabla \nu^u$ is measurable and such that $q^u |\nabla \nu^u| \in L^2(A)$ and

$$(2.12) \quad \int_A |\nabla u|^2 dx = \int_A |\nabla q^u|^2 dx + \int_A (q^u)^2 |\nabla \nu^u|^2 dx.$$

Moreover for $q^h \in W^{1,2}(A) \cap L^\infty(A)$, $q^u \geq 0$ the vector function σ defined via

$$(2.13) \quad \sigma = a + q^\sigma \nu^u, \quad q^\sigma = \min\{q^h, q^u\}$$

is in $W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ and satisfies the corresponding (2.12).

By the polar form (2.12) of the energy and the minimality of u assumed in (HB) it follows that

$$(2.14) \quad \begin{aligned} & \frac{1}{2} \int_{B_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dx \\ &= J_{B_R}(u) - J_{B_R}(\sigma) + \frac{1}{2} \int_{B_R} \left((q^\sigma)^2 - (q^u)^2 \right) |\nabla \nu^u|^2 dx + \int_{B_R} (W(\sigma) - W(u)) dx \\ &\leq \int_{B_R} (W(\sigma) - W(u)) dx \end{aligned}$$

where we have also used the definition (2.13) of σ which implies $q^\sigma \leq q^u$.

2. The Isoperimetric Inequality for Minimizers

We will assume that $q^h \geq q^u$ on ∂B_R and therefore by (2.13) that $q^\sigma = q^u$ on ∂B_R , q^h to be further specified later. Define

$$(2.15) \quad \begin{cases} A_r = \int_{B_r \cap \{q^u \leq \lambda\}} W(u) dx, \\ V_r = \mathcal{L}^n(B_r \cap \{q^u > \lambda\}). \end{cases}$$

We also define the cut-off function

$$(2.16) \quad \beta = \min\{q^u - q^\sigma, \lambda\}, \quad \text{on } B_R, \quad \lambda > 0 \text{ small.}$$

which is related via the map $a + \beta \nu^u$ to the variation σ in (2.13). The modification in the definition of A with the integration over the sub-level set together with the definition of the function β in the context of the Caffarelli-Cordoba [14] set-up was introduced in Valdinoci [28]. By applying the inequality in [16] pag.141 to β^2 we obtain

$$(2.17) \quad \begin{aligned} \left(\int_{B_R} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} &= \left(\int_{B_R} (\beta^2)^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} dx \\ &\leq C \int_{B_R} |\nabla(\beta^2)| dx \leq 2C \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta| |\beta| dx, \end{aligned}$$

where $C > 0$ is a constant independent of R and we have used $\beta = 0$ on ∂B_R and the fact that $\nabla \beta = 0$ a.e. on $q^u - q^\sigma > \lambda$. By Young's inequality, for $A > 0$ we have

$$(2.18) \quad \begin{aligned} & \left(\int_{B_R} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq 2C \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta| |\beta| dx \\ &\leq CA \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta|^2 dx + \frac{C}{A} \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} \beta^2 dx \\ &\leq CA \int_{B_R} |\nabla(q^u - q^\sigma)|^2 dx + \frac{C}{A} \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx \\ &= CA \left(\int_{B_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dx - 2 \int_{B_R} \nabla q^\sigma \cdot \nabla(q^u - q^\sigma) dx \right) \\ &+ \frac{C}{A} \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx. \end{aligned}$$

From (2.18) and (2.14) it follows

$$\begin{aligned}
(2.19) \quad & \left(\int_{B_R} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \\
& \leq 2CA \left(\int_{B_R} (W(\sigma) - W(u)) dx - \int_{B_R} \nabla q^\sigma \cdot \nabla (q^u - q^\sigma) dx \right) \\
& + \frac{C}{A} \int_{B_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx.
\end{aligned}$$

3. The case $0 < \alpha < 2$.

Assume that $q^h \in W^{1,2}(B_R) \cap L^\infty(B_R)$ satisfies

$$(2.20) \quad q^h = 0, \quad \text{on } B_{R-T} \text{ for some fixed } T > 0.$$

The Lower Bound

From (2.20) it follows

$$(2.21) \quad \left(\int_{B_R} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \geq \left(\int_{B_{R-T} \cap \{q^u > \lambda\}} \beta^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} dx \geq \lambda^2 \mathcal{L}^n(B_{R-T} \cap \{q^u > \lambda\})^{\frac{n-1}{n}}$$

where we have also used (2.20) which implies $q^\sigma = 0$ on B_{R-T} .

The Upper Bound

The objective is to estimate the right and side of (2.19) by the first term involving the potential. Naturally the third term can be handled more easily for $\alpha < 2$. For handling the second term one needs a very particular choice of q^h . The splitting of the integrations over B_{R-T} and the rest aims at deriving a difference inequality involving the quantities in (2.15), as in (2.33). A major difference between $\alpha < 2$ and $\alpha = 2$ is in the choice of q^h , that can vanish on B_{R-T} for $\alpha < 2$, while can only be exponentially small (in T) for $\alpha = 2$.

We begin with B_{R-T} .

Since $q^\sigma = 0$ on B_{R-T} the right hand side I of (2.19) on B_{R-T} reduces to

$$\begin{aligned}
(2.22) \quad I &= -2CA \int_{B_{R-T}} W(u) dx + \frac{C}{A} \int_{B_{R-T} \cap \{q^u \leq \lambda\}} (q^u)^2 dx \\
&\leq -2CA \int_{B_{R-T} \cap \{q^u \leq \lambda\}} W(u) dx + \frac{C}{A} \int_{B_{R-T} \cap \{q^u \leq \lambda\}} (q^u)^2 dx
\end{aligned}$$

Claim 1

Assume $\lambda \leq \rho_0$, ρ_0 the constant in (HA). Then there exists $A_0 > 0$ independent of R such that

$$(2.23) \quad I \leq -CA \int_{B_{R-T} \cap \{q^u \leq \lambda\}} W(u) dx, \quad \text{for } A > A_0.$$

Proof. From (HA) $q^u \leq \lambda \leq \rho_0$ it follows

$$\begin{aligned}
W(u) &= \int_0^{q^u} W_u(a + s\nu^u) \cdot \nu^u ds \geq \frac{C^*}{\alpha} (q^u)^\alpha \\
\text{hence } -AW(u) + \frac{1}{A} (q^u)^2 &\leq (q^u)^\alpha \left(\frac{-AC^*}{\alpha} + \frac{\lambda^{2-\alpha}}{A} \right)
\end{aligned}$$

and therefore for $A > \sqrt{\alpha\lambda^{2-\alpha}/C^*}$ we obtain

$$-CA \int_{B_{R-T} \cap \{q^u \leq \lambda\}} W(u) dx + \frac{C}{A} \int_{B_{R-T} \cap \{q^u \leq \lambda\}} (q^u)^2 dx \leq 0.$$

This and (2.22) conclude the proof of the claim. \square

Next we consider the right hand side of (2.19) on $B_R \setminus B_{R-T}$.
Set

$$\begin{aligned} I_1 &= 2CA \int_{B_R \setminus B_{R-T}} (W(\sigma) - W(u)) dx + \frac{C}{A} \int_{(B_R \setminus B_{R-T}) \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dx, \\ I_2 &= -2CA \int_{B_R \setminus B_{R-T}} \nabla q^\sigma \cdot \nabla (q^u - q^\sigma) dx. \end{aligned}$$

Claim 2

Assume $\lambda \leq \min\{\rho_0, 1\}$. Then there exists constant $\tilde{C} > 0$ independent of R such that

$$(2.24) \quad I_1 \leq \tilde{C} A \mathcal{L}^n((B_R \setminus B_{R-T}) \cap \{q^u > \lambda\}) + \frac{\tilde{C}}{A} \int_{(B_R \setminus B_{R-T}) \cap \{q^u \leq \lambda\}} W(u) dx, \text{ for } A > 0.$$

Proof. We split the integration in $B_R \setminus B_{R-T}$ over $\{q^u \leq \lambda\}$ and $\{q^u > \lambda\}$. From $q^\sigma \leq q^u$, $q^u \leq \lambda \leq \rho_0$ we have

$$\int_{(B_R \setminus B_{R-T}) \cap \{q^u \leq \lambda\}} (W(\sigma) - W(u)) dx \leq 0$$

and therefore from (2.5) it follows

$$(2.25) \quad \int_{B_R \setminus B_{R-T}} (W(\sigma) - W(u)) dx \leq W_M \mathcal{L}^n((B_R \setminus B_{R-T}) \cap \{q^u > \lambda\})$$

where $W_M = \max_{|u-a| \leq M} W(u)$. As in the proof of Claim 1, for $q^\sigma \leq q^u \leq \lambda \leq \min\{\rho_0, 1\}$, we get

$$W(u) \geq \frac{C^*}{\alpha} (q^u)^\alpha \geq \frac{C^*}{\alpha} (q^u - q^\sigma)^\alpha \geq \frac{C^*}{\alpha} (q^u - q^\sigma)^2$$

which implies

$$\int_{(B_R \setminus B_{R-T}) \cap \{q^u \leq \lambda\}} (q^u - q^\sigma)^2 dx \leq \frac{\alpha}{C^*} \int_{(B_R \setminus B_{R-T}) \cap \{q^u \leq \lambda\}} W(u) dx.$$

This and (2.25) establish Claim 2 with $\tilde{C} = C \max\{\alpha/C^*, 2W_M, M^2\}$. \square

We now complete the definition (2.20) of q^h by setting as in [14]

$$(2.26) \quad q^h(x) = H(|x| - (R - T))^{\frac{2}{2-\tau}}, \text{ on } B_R \setminus B_{R-T}$$

where $\tau = \max\{\alpha, 1\}$ and $H = M/T^{\frac{2}{2-\tau}}$ is chosen so that $q^h = M$ on ∂B_R . Note that q^h is C^1 on B_R and

$$(2.27) \quad \nabla q^h = \nabla q^\sigma = 0 \text{ on } \partial B_{R-T}$$

where we have also used that $q^\sigma \leq q^h$. The function $[0, T] \ni s \mapsto q(s) = Hs^{\frac{2}{2-\tau}}$ satisfies

$$(2.28) \quad q'' = c_H q^{\tau-1}, \quad q' = \sqrt{2c_H/\tau} q^{\frac{\tau}{2}}$$

where c_H is a constant that depends on H . Since $\tau < 2$ implies $\tau - 1 < \frac{\tau}{2}$, (2.28) yields

$$(2.29) \quad \Delta q^h \leq C_1 (q^h)^{\tau-1}$$

with $C_1 > 0$ independent of R .

Claim 3

There exists $\hat{C} > 0$ independent of R such that

$$(2.30) \quad \begin{aligned} I_2 &\leq \hat{C} A \mathcal{L}^n((B_R \setminus B_{R-T}) \cap \{q^u > \lambda\}) \\ &+ \hat{C} A \int_{(B_R \setminus B_{R-T}) \cap \{q^u \leq \lambda\}} W(u) dx, \quad \text{for } A > 0. \end{aligned}$$

Proof. From (2.27) and $q^u = q^\sigma$ on ∂B_R and integration by parts it follows

$$(2.31) \quad I_2 = 2CA \int_{(B_R \setminus B_{R-T})} \Delta q^\sigma (q^u - q^\sigma) dx = 2CA \int_{(B_R \setminus B_{R-T}) \cap \{q^h < q^u\}} \Delta q^h (q^u - q^h) dx$$

where we have observed that $q^u = q^\sigma$ on the set $\{q^h \geq q^u\}$ and that $q^\sigma = q^h$ on the set $\{q^h < q^u\}$. From (2.29) and $q^h \leq q^u$ it follows

$$(2.32) \quad \begin{aligned} I_2 &\leq 2CC_1 A \int_{B_R \setminus B_{R-T} \cap \{q^h < q^u\}} (q^h)^{\tau-1} (q^u - q^h) dx \\ &\leq 2CC_1 A \int_{B_R \setminus B_{R-T} \cap \{q^h < q^u\}} (q^u)^\tau dx. \end{aligned}$$

As before we split the integration over $\{q^u \leq \lambda\}$ and $\{q^u > \lambda\}$. To conclude the proof we observe that $\lambda \leq \min\{\rho_0, 1\}$ and (HA) imply

$$(q^u)^\tau \leq (q^u)^\alpha \leq \frac{\alpha}{C_*} W(u), \quad \text{on } \{q^u \leq \lambda\}$$

while (2.5) implies

$$(q^u)^\tau \leq M^\tau, \quad \text{on } \{q^u > \lambda\}.$$

□

We are now in the position of completing the proof of Theorem A for the case $0 < \alpha < 2$. By recalling the definition of A_R and V_R in (2.15) and by collecting all the estimates (2.21), (2.23), (2.24) and (2.30) we have for fixed $A > A_0$

$$\lambda^2 (V_{R-T})^{\frac{n-1}{n}} + CA A_{R-T} \leq (\tilde{C} + \hat{C}) A (V_R - V_{R-T}) + \left(\frac{\tilde{C}}{A} + \hat{C}\right) (A_R - A_{R-T})$$

and consequently

$$(2.33) \quad C(\lambda) \left((V_{R-T})^{\frac{n-1}{n}} + V_{R-T} \right) \leq (V_R - V_{R-T}) + (A_R - A_{R-T})$$

with $C(\lambda) = \frac{\min\{\lambda^2, CA\}}{\max\{(\tilde{C}+\tilde{C})A, \frac{C}{A}+\tilde{C}A\}}$. Equation (2.33) is exactly the difference scheme in [14]. Therefore as in [14], using also the assumption (2.6), we deduce that there are $C(\lambda, \mu_0) > 0$ and $k_0 \geq 1$ such that

$$(2.34) \quad V_{kT} + A_{kT} \geq C(\lambda, \mu_0)k^n, \quad \text{for } k \geq k_0.$$

To complete the argument we recall the basic estimate (2.35) below (c.f. Lemma 1 in [14] for the scalar case. The proof is similar for the vector case)

Lemma 2.2. *Assume that W satisfies (HA) and assume that u is minimal as defined in (HB). Then there is a constant $C > 0$, depending on M , independent of ξ and such that*

$$(2.35) \quad \int_{B_R(\xi)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \leq CR^{n-1}, \quad \text{for } R > 0.$$

From (2.35) we obtain $A_{kT} \leq C(kT)^{n-1}$. This concludes the proof of Theorem A in the case $0 < \alpha < 2$ for $\lambda > 0$ small. The restriction on the smallness of λ is easily removed via (2.35).

4. The case $\alpha = 2$.

We let $\varphi : B_R \rightarrow \mathbb{R}$ the solution of the problem

$$(2.36) \quad \begin{cases} \Delta \varphi = c_1 \varphi, & \text{on } B_R, \\ \varphi = 1, & \text{on } \partial B_R, \end{cases}$$

where $c_1 < c_0$ will be chosen later and c_0 is the constant in (HA). It is well known that φ satisfies the exponential estimate

$$(2.37) \quad \varphi(R-r) \leq e^{-c_2(R-r)}, \quad \text{for } r \in [0, R], \quad R \geq 1,$$

for some $c_2 > 0$.

Define

$$(2.38) \quad q^h = \varphi M,$$

and as before

$$(2.39) \quad \begin{aligned} q^\sigma &= \min\{q^u, q^h\}, \\ \beta &= \min\{q^u - q^\sigma, \lambda\}, \end{aligned}$$

From (2.19), $q^\sigma = q^u$ on ∂B_R , and an integration by parts we get

$$(2.40) \quad \begin{aligned} & \left(\int_{B_R} \beta^{\frac{2n}{n-1}} \right)^{\frac{n-1}{n}} \\ & \leq 2CA \int_{B_R} \left(W(\sigma) - W(u) + \Delta q^\sigma (q^u - q^\sigma) \right) dx + \frac{C}{A} \int_{B_R \cap \{q^u - q^\sigma < \lambda\}} (q^u - q^\sigma)^2 dx \\ & = 2CA \int_{B_R \cap \{q^u > q^h\}} \left(W(h) - W(u) + \Delta q^h (q^u - q^h) \right) dx + \frac{C}{A} \int_{B_R \cap \{0 < q^u - q^h < \lambda\}} (q^u - q^h)^2 dx, \end{aligned}$$

where we have used that $q^u > q^\sigma$ implies $q^\sigma = q^h$, $h = a + q^h \nu^u$. By (HA) there is $\lambda^* > \lambda$ sufficiently small (and fixed from now on) so that the maps $s \mapsto W(a + s\nu)$ and $s \mapsto W_u(a + s\nu) \cdot \nu$ are increasing in $[0, \lambda^*]$.

Claim 4

$$\begin{aligned}
(2.41) \quad & \left(\int_{B_R} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
& \leq 2CA \int_{B_R \cap \{q^u > q^h\} \cap \{q^u > \lambda^*\}} \left(W(h) - W(u) + \Delta q^h(q^u - q^h) \right) dx \\
& \quad + \frac{C}{A} \int_{B_R \cap \{0 < q^u - q^h < \lambda\} \cap \{q^u > \lambda^*\}} (q^u - q^h)^2 dx.
\end{aligned}$$

Proof. In $B_R \cap \{q^u \leq \lambda^*\}$ we have

$$\begin{aligned}
(2.42) \quad & W(u) - W(h) = \int_{q^h}^{q^u} W_u(a + s\nu) \cdot \nu ds \geq \int_{q^h}^{q^u} c_0 s ds = \frac{1}{2} c_0 (q^u + q^h)(q^u - q^h), \\
& \Delta q^h(q^u - q^h) = c_1 q^h (q^u - q^h),
\end{aligned}$$

where we have also utilized (2.36), (2.38) and (2.39). From (2.42) it follows

$$\begin{aligned}
(2.43) \quad & 2CA(W(h) - W(u) + \Delta q^h(q^u - q^h)) + \frac{C}{A}(q^u - q^h)^2 \\
& \leq (2CA(-\frac{1}{2}c_0(q^u + q^h) + c_1 q^h) + \frac{C}{A}(q^u - q^h))(q^u - q^h).
\end{aligned}$$

For $c_1 > 0$ small and $A > 0$ large ($c_1 \leq \frac{1}{4}c_0$ and $A \geq \sqrt{\frac{2}{c_0}}$) the last expression in (2.43) is negative. Therefore we also have

$$\begin{aligned}
(2.44) \quad & 2CA \int_{B_R \cap \{q^u > q^h\} \cap \{q^u \leq \lambda^*\}} \left(W(h) - W(u) + \Delta q^h(q^u - q^h) \right) dx \\
& \quad + \frac{C}{A} \int_{B_R \cap \{0 < q^u - q^h < \lambda\} \cap \{q^u \leq \lambda^*\}} (q^u - q^h)^2 dx \leq 0.
\end{aligned}$$

This and (2.40) conclude the proof of Claim 4. \square

Set $R = (k+1)T$ where $T > 0$ is a large number to be chosen later. Set

$$(2.45) \quad \omega_j = \mathcal{L}^n((B_{jT} \setminus B_{(j-1)T}) \cap \{q^u > \lambda^*\}), \quad j = 1, \dots, k+1.$$

Claim 5

$$(2.46) \quad C_0 \left(\sum_{j=1}^k \omega_j \right)^{\frac{n-1}{n}} \leq \sum_{j=1}^k e^{-c_2 j T} \omega_{k+1-j} + \omega_{k+1}, \quad k = 1, \dots$$

where c_2 is the constant in (2.37) and $C_0 > 0$ is a constant, $C_0 = C_0(A, \lambda, M)$.

Proof. On B_{kT} we have $q^h \leq M e^{-c_2 T}$ and therefore we can choose $T > 0$ so large that

$$\begin{aligned}
(2.47) \quad & x \in B_{kT} \cap \{q^u > \lambda^*\} \Rightarrow q^u - q^h \geq \lambda^* - M e^{-c_2 T} > \lambda \\
& \Rightarrow B_{kT} \cap \{q^u - q^h < \lambda\} \cap \{q^u > \lambda^*\} = \emptyset.
\end{aligned}$$

We begin by estimating part of the right hand side of (2.41) over $B_R \setminus B_{R-T}$ by utilizing (2.47) and (2.32)

$$\begin{aligned}
(2.48) \quad & 2CA \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{q^u > q^h\} \cap \{q^u > \lambda^*\}} \left(W(h) - W(u) + \Delta q^h(q^u - q^h) \right) dx \\
& + \frac{C}{A} \int_{B_{(k+1)T} \cap \{0 < q^u - q^h < \lambda\} \cap \{q^u > \lambda^*\}} (q^u - q^h)^2 dx \\
& \leq 2CA \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{q^u > q^h\} \cap \{q^u > \lambda^*\}} \left(W(h) + \Delta q^h(q^u - q^h) \right) dx \\
& + \frac{C}{A} \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{0 < q^u - q^h < \lambda\} \cap \{q^u > \lambda^*\}} (q^u - q^h)^2 dx \\
& \leq (2CA(\overline{W} + c_1 M^2) + \frac{C}{A} \lambda^2) \mathcal{L}^n((B_{(k+1)T} \setminus B_{kT}) \cap \{q^u > \lambda^*\}) \\
& = C^* \omega_{k+1}
\end{aligned}$$

where we have set $\overline{W} = \max_{|u-a| \leq M} W(u)$ and $C^* = 2CA(\overline{W} + c_1 M^2) + \frac{C}{A} \lambda^2$.

Next we estimate the remaining part of (2.41) over B_{R-T} . The smoothness of W implies that there are $C_0 > 0$ and $\bar{q} > 0$ such that

$$(2.49) \quad W(a + s\nu) \leq \frac{1}{2} C_0 s^2, \quad \text{for } s \in [0, \bar{q}].$$

We can assume $T > 0$ so large that $Me^{-c_2 T} \leq \bar{q}$. Then we have

$$\begin{aligned}
(2.50) \quad & x \in ((B_{(k+1-j)T} \setminus B_{(k-j)T}) \cap \{q^u > \lambda^*\} \cap \{q^u > q^h\}) \\
& \Rightarrow W(h) + \Delta q^h(q^u - q^h) \leq M^2 e^{-c_2 j T} \left(\frac{1}{2} C_0 e^{-c_2 j T} + c_1 \right) \\
& \Rightarrow 2CA \int_{(B_{(k+1-j)T} \setminus B_{(k-j)T}) \cap \{q^u > \lambda^*\} \cap \{q^u > q^h\}} (W(h) + \Delta q^h(q^u - q^h)) dx \\
& \leq 2CAM^2 \left(\frac{1}{2} C_0 + c_1 \right) e^{-c_2 j T} \omega_{k+1-j} = C^\circ \omega_{k+1-j} \epsilon^j
\end{aligned}$$

where we have set $C^\circ = 2CAM^2(\frac{1}{2}C_0 + c_1)$ and $\epsilon = e^{-c_2 T}$. From (2.50) we obtain

$$(2.51) \quad 2CA \int_{B_{kT} \cap \{q^u > \lambda^*\} \cap \{q^u > q^h\}} (W(h) - W(u) + \Delta q^h(q^u - q^h)) dx \leq C^\circ \sum_{j=1}^k \epsilon^j \omega_{k+1-j}.$$

Combining (2.51), (2.48) in (2.41) we obtain the upper bound

$$(2.52) \quad \left(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq C^\circ \sum_{j=1}^k \epsilon^j \omega_{k+1-j} + C^* \omega_{k+1}$$

To estimate the left hand side of (2.52) from below we observe that (2.47) implies

$$\begin{aligned}
& \left(\int_{B_{kT} \cap \{q^u - q^h < \lambda\} \cap \{q^u > \lambda^*\}} (q^u - q^h)^{\frac{2n}{n-1}} dx + \int_{B_{kT} \cap \{q^u - q^h \geq \lambda\} \cap \{q^u > \lambda^*\}} \lambda^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&= \left(\int_{B_{kT} \cap \{q^u > \lambda^*\}} \lambda^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
(2.53) \quad &= \lambda^2 \left(\sum_{j=1}^k \omega_j \right)^{\frac{n-1}{n}} \\
&= \left(\int_{B_{kT} \cap \{q^u > \lambda^*\}} (\beta^2)^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \\
&\leq \left(\int_{B_{(k+1)T}} \beta^{\frac{2n}{n-1}} dx \right)^{\frac{n-1}{n}}.
\end{aligned}$$

Combining this with (2.52) we obtain (2.46). The proof of Claim 5 is complete \square

Claim 6

From (2.46) it follows

$$(2.54) \quad \omega_k \geq c^* k^{n-1}, \text{ for } k = 1, 2, \dots$$

for some $c^* > 0$. Then (2.54) implies

$$\mathcal{L}^n(B_R \cap \{q^u > \lambda^*\}) \geq \frac{c^*}{n 2^n T^n} R^n, \text{ for } R \geq T.$$

Proof. We proceed by induction. For $k = 1$ (2.54) holds by (2.6) for any $0 < c^* \leq \mu_0$, $T \geq 1$. Thus we assume that (2.54) holds true for $j \leq k$ and show that it is true for $k+1$. From the inductive assumption we have

$$(2.55) \quad \frac{c^*}{n} k^n = c^* \int_0^k j^{n-1} dj \leq c^* \sum_{j=1}^k j^{n-1} \leq \sum_{j=1}^k \omega_j.$$

Therefore for the left hand side of (2.46) we have the lower bound

$$(2.56) \quad \frac{C_0}{2^n} \left(\frac{c^*}{n} \right)^{\frac{n-1}{n}} (k+1)^{n-1} \leq C_0 \left(\frac{c^*}{n} \right)^{\frac{n-1}{n}} k^{n-1}.$$

Observe now that we have the obvious bound

$$(2.57) \quad \omega_j \leq \eta j^{n-1} T^n,$$

where η is the measure of the unit sphere in \mathbb{R}^n . Therefore we can derive for the right hand side of (2.46) the upper bound

$$(2.58) \quad \sum_{j=1}^k \epsilon^j \omega_{k+1-j} + \omega_{k+1} \leq \eta T^n k^{n-1} \sum_{j=1}^k \epsilon^j + \omega_{k+1} \leq \eta T^n k^{n-1} \frac{\epsilon}{1-\epsilon} + \omega_{k+1}$$

From this and (2.56) we get

$$(2.59) \quad \frac{C_0}{2^n} \left(\frac{c^*}{n}\right)^{\frac{n-1}{n}} (k+1)^{n-1} \leq \eta T^n \frac{\epsilon}{1-\epsilon} k^{n-1} + \omega_{k+1}.$$

Since $\epsilon = e^{-c_2 T}$ we can choose $T > 0$ so large that

$$\eta T^n \frac{\epsilon}{1-\epsilon} \leq \frac{C_0}{2^{n+1}} \left(\frac{c^*}{n}\right)^{\frac{n-1}{n}}.$$

Then from (2.59) we obtain

$$(2.60) \quad \frac{C_0}{2^{n+1}} \left(\frac{c^*}{n}\right)^{\frac{n-1}{n}} (k+1)^{n-1} \leq \omega_{k+1}.$$

Therefore to complete the induction it suffices to observe that we can choose c^* so small that

$$c^* \leq \frac{C_0}{2^{n+1}} \left(\frac{c^*}{n}\right)^{\frac{n-1}{n}} \Leftrightarrow 1 \leq \frac{C_0}{2^{n+1} n^{\frac{n-1}{n}} (c^*)^{\frac{1}{n}}}.$$

Let $[R/T]$ the integer part of R/T and observe that

$$\frac{[R/T]}{R/T} \geq \frac{1}{2}, \quad \text{for } R \geq T.$$

From (2.54) and (2.55) we have

$$\mathcal{L}^n(B_R \cap \{q^u > \lambda^*\}) \geq \sum_{k=1}^{[R/T]} \omega_k \geq \frac{c^*}{n T^n} \left(\frac{[R/T]}{R/T}\right)^n R^n \geq \frac{c^*}{n 2^n T^n} R^n, \quad \text{for } R \geq T.$$

□

Claim 6 concludes the case $\alpha = 2$ and completes the proof of Theorem A for small $\lambda > 0$. As in the case $\alpha < 2$ the restriction on the smallness of λ is removed via (2.35).

3 Pointwise Estimates-Liouville type results

Theorem A implies the following basic estimate (cfr. Theorem 1.2 in Fusco [20])

Theorem 3.1. *Assume that W satisfies (HA) and assume that $u : D \rightarrow \mathbb{R}^m$ is minimal in the sense of (HB), $D \subset \mathbb{R}^n$ open. Let $\mathcal{Z} := \{W = 0\} \setminus \{a\}$ and assume*

$$(3.1) \quad \mathcal{Z} = \emptyset \quad \text{or} \quad d_0 = \inf_{x \in D} d(u(x), \mathcal{Z}) > 0, \quad d \text{ the Euclidean distance}.$$

Then, given $\lambda > 0$, there is $R(\lambda)$ such that

$$(3.2) \quad B_{R(\lambda)}(x_0) \subset D, \Rightarrow |u(x_0) - a| < \lambda.$$

$R(\lambda)$ depends only on W and on the bound M in (HB) if $\mathcal{Z} = \emptyset$ and also on d_0 otherwise.

Proof. Let $R_{x_0} = \max\{R : B_R(x_0) \subset D\}$ and assume $R(x_0) > 1$. Then, from (2.5), we have that the inequality

$$|u(x_0) - a| \geq \lambda$$

implies

$$\mathcal{L}^n(B_1(x_0) \cap \{|u(x) - a| \geq \frac{\lambda}{2}\}) \geq \mu_0 > 0$$

and therefore Theorem A yields

$$(3.3) \quad \mathcal{L}^n(B_R(x_0) \cap \{|u(x) - a| \geq \frac{\lambda}{2}\}) \geq \tilde{C}R^n, \quad \text{for } 1 < R < R_{x_0}$$

and a constant $\tilde{C} = \tilde{C}(\lambda, M) > 0$ independent of x_0 . Observe that the assumption (3.1) implies via (3.3)

$$(3.4) \quad \bar{w}\tilde{C}R^n \leq \int_{B_R(x_0)} W(u)dx \leq J_{B_R(x_0)}, \quad \text{for } R \leq R_{x_0}$$

where we have set

$$\bar{w} = \min\{W(z) : |z - a| \geq \frac{\lambda}{2}, d(z, \mathcal{Z}) \geq d_0, |z - a| \leq M\} > 0.$$

The inequality (3.4) and the upper bound (2.35) in Lemma 2.2 are compatible only if $R \leq \frac{C}{\bar{w}\tilde{C}}$ where C is the constant in Lemma 2.2. Therefore if

$$R_{x_0} \geq 2\frac{C}{\bar{w}\tilde{C}}$$

we necessarily have

$$|u(x_0) - a| < \lambda.$$

This concludes the proof with $R(\lambda) = 2\frac{C}{\bar{w}\tilde{C}}$. \square

Theorem 3.1 allows to extend to potentials that satisfy (HA) and in particular to singular potentials ($\alpha \in (0, 1]$) the following *Liouville* type result established in [20].

Theorem 3.2. *Let W and u be as in Theorem 3.1 and assume $D = \mathbb{R}^n$. Then*

$$u \equiv a.$$

Proof. $D = \mathbb{R}^n$ trivially implies that, given $x_0 \in \mathbb{R}^n$ and $\lambda > 0$, $B_{R(\lambda)}(x_0) \subset D$. Then Theorem 3.1 yields

$$|u(x_0) - a| < \lambda, \quad \text{for } \lambda > 0, x_0 \in \mathbb{R}^n.$$

The proof is complete. \square

The following exponential estimate (see [20] Theorem 1.3) can be considered a consequence of the density estimate in Theorem A.

Theorem 3.3. *Let $u : D \rightarrow \mathbb{R}^m$ and W as in Theorem 3.1. Assume $\alpha = 2$ in (HA) and $D \neq \mathbb{R}^n$ with $\sup_{x_0 \in D} R_{x_0} = +\infty$. Then*

$$|u(x) - a| \leq Ke^{-kd(x, \partial D)}, \quad \text{for some } k, K > 0.$$

Proof. First we note that it is sufficient to establish that, given a small number $\lambda > 0$, there is $d_\lambda > 0$ such that

$$d(x, \partial D) \geq d_\lambda \Rightarrow |u(x) - a| \leq \lambda$$

since then linear theory renders the result. From Theorem 3.1 it follows that we can take $d_\lambda = R(\lambda)$. The proof is complete. \square

4 On the Linking with the Minimal Surface Problem

We will consider partitions with Dirichlet conditions for simplicity. The volume constraint case is more involved but similar. Assume that W is as in Theorem 2.1 and that therefore

$$0 = W(a_1) = W(a_2) < W(u), \quad u \notin \{a_1, a_2\}$$

for $a_1 \neq a_2 \in \mathbb{R}^m$. Let $\{u_{\epsilon_k}\}$ be a sequence of global minimizers of $J_D^\epsilon(u) = \int_D (\frac{\epsilon^2}{2} |\nabla u|^2 + W(u)) dy$ subject to the Dirichlet condition $u_{\epsilon_k} = g$ on ∂D , $g : \partial D \rightarrow \{a_1, a_2\}$.

We assume $D \subset \mathbb{R}^n$ open bounded with C^1 boundary and consider a partition of the boundary $B_j = g^{-1}(\{a_j\})$, $j = 1, 2$ with $\mathcal{H}^{n-1}(\partial D \setminus (B_1 \cup B_2)) = 0$. We also assume that $\|u_{\epsilon_k}\|_{L^\infty(D; \mathbb{R}^m)} < M$ uniformly. Then by the methods in Baldo [10] $\{u_{\epsilon_k}\}$ is relatively compact in $L^1(D; \mathbb{R}^m)$ and along a subsequence $\epsilon_k \rightarrow 0^+$ $u_{\epsilon_k} \xrightarrow{L^1} u_0 = a_1 \chi_{D_1} + a_2 \chi_{D_2}$ where D_1, D_2 is a partition of D with $\partial D_j \cap \partial D = B_j$, $j = 1, 2$. Moreover the interface $\partial D_1 \cap \partial D_2$ minimizes $\mathcal{H}^{n-1}(\partial A_1 \cap \partial A_2)$ among all partitions of D with Dirichlet conditions B . For two-phase partitions, if $n \leq 7$, the interface $\partial D_1 \cap \partial D_2$ is locally a real analytic classical minimal surface (see [21]).

We write u_ϵ in polar form (cfr. (2.10)), $u_\epsilon = a_1 + \rho_\epsilon \nu_\epsilon$ with

$$\rho_\epsilon(y) = |u_\epsilon - a_1|, \quad \nu_\epsilon(y) = \frac{u_\epsilon - a_1}{|u_\epsilon - a_1|}, \quad \nu_\epsilon(y) = 0 \quad \text{if } \rho_\epsilon(y) = 0.$$

Then, from $u_{\epsilon_k} \xrightarrow{L^1} u_0$, we obtain that

$$(4.1) \quad \rho_{\epsilon_k} \rightarrow \rho_0 = \begin{cases} 0 & \text{in } D_1, \\ |a_1 - a_2| & \text{in } D_2. \end{cases}$$

Proposition 4.1. *The level set $\mathcal{S}_\epsilon = \{y \in D : |u_\epsilon - a_j| = \gamma, j = 1, 2\}$, $\gamma \in (0, |a_1 - a_2|)$ converges locally uniformly to $\partial D_1 \cap \partial D_2$ as $\epsilon \rightarrow 0^+$.*

Proof. (Blow-up, cfr. Theorem 2 in [14]) Suppose that the convergence is not uniform over a compact set $\mathcal{K} \subset \subset D$. Then there are sequences $\epsilon_k \rightarrow 0^+$, $y_k \in \mathcal{S}_{\epsilon_k} \cap \mathcal{K}$, $k = 1, \dots$ and $r > 0$ such that $d(y_k, \partial D_1 \cap \partial D_2) \geq r$. We can assume that all the points y_k are in one of the sets D_j , $j = 1, 2$. For definiteness we suppose $y_k \in D_1$, $k = 1, \dots$. Actually $\mathcal{K} \subset \subset D$ implies that we can assume $B_r(y_k) \subset D_1$, $k = 1, \dots$. Set

$$x = \frac{y - y_k}{\epsilon_k}, \quad v_k = u_{\epsilon_k}(\epsilon_k x + y_k), \quad \varrho_k(x) = \rho_{\epsilon_k}(\epsilon_k x + y_k).$$

Since u_{ϵ_k} is a minimizer we have $\Delta v_k - W(v_k) = 0$, $\varrho_k(0) = \gamma$ and $|v_k - a| < M$. Thus we also have the gradient bound $|\nabla v_k| < M$ which implies

$$\varrho_k(x) > \frac{\gamma}{2} \quad \text{for } |x| < \delta$$

for some $\delta > 0$ independent of $k = 1, \dots$. Now we observe that v_k , $k = 1, \dots$ is a minimizer of $J_{D_k}(v) = \int_{D_k} (\frac{1}{2} |\nabla v|^2 + W(v)) dx$, $D_k = \{x = y - y_k/\epsilon_k, y \in D\}$. Thus we can apply Theorem A that yields the density estimate

$$(4.2) \quad \mathcal{L}^n(\{|x| < R\} \cap \{\varrho_k(x) > \frac{\gamma}{2}\}) \geq CR^n, \quad R \geq \delta$$

that holds uniformly over the family $\{v_k\}$. This estimate is equivalent to

$$\mathcal{L}^n(B_{\epsilon_k R}(y_k) \cap \{\rho_{\epsilon_k}(y) > \frac{\gamma}{2}\}) \geq C(\epsilon_k R)^n, \quad R \geq \delta.$$

In particular, for $R = r/\epsilon_k$, we get

$$(4.3) \quad \mathcal{L}^n(B_r(y_k) \cap \{\rho_{\epsilon_k}(y) > \frac{\gamma}{2}\}) \geq Cr^n.$$

Since $B_r(y_k) \subset D_1$ and $\rho_0 = 0$ a.e. on D_1 (4.3) implies

$$\int_D |\rho_{\epsilon_k} - \rho_0| dy \geq \int_{B_r(y_k)} \rho_{\epsilon_k} dy \geq \frac{\gamma}{2} Cr^n$$

which contradicts (4.1). The proof is complete. \square

5 A Lower Bound for the Energy

In this section we adopt the following hypothesis

(HC) There exists $N \geq 2$ and N distinct points $a_1, \dots, a_N \in \mathbb{R}^m$ such that

$$0 = W(a_j) < W(u), \quad j = 1, \dots, N, \quad u \in \mathbb{R}^m \setminus \{a_1, \dots, a_N\}.$$

Moreover $W : \mathbb{R}^m \rightarrow \mathbb{R}$ is as in (HA) for $a = a_j$, $j = 1, \dots, N$.

From the monotonicity formula (see (1.4) in [2]), which holds for general Lipschitz $W \geq 0$, it follows that *any* solution to $\Delta u - W_u(u) = 0$ satisfies the lower bound

$$(5.1) \quad J_{B_R(x_0)}(u) \geq R^{n-2} J_{B_1(x_0)}(u), \quad R \geq 1.$$

If $W(u) = (1 - |u|^2)^2$ and, more generally, if the set of the zeros of W is not totally disconnected, the lower bound above is sharp (see (2.4) in Farina [17]). On the other hand for the class of phase transition potentials defined in (HC) above, under the hypothesis of minimality we have

Proposition 5.1. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be nonconstant and minimal in the sense of (HB), and pointwise bounded uniformly over \mathbb{R}^n (cfr. (2.5)). Then we have*

$$(5.2) \quad \int_{B_R(x_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \geq CR^{n-1}, \quad R \geq R(x_0).$$

with $C > 0$ independent of x_0 .

Proof. Since u is continuous and nonconstant there are $\gamma > 0$ and $\xi \in \mathbb{R}^n$ such that $|u(\xi) - a_j| > \gamma$, $j = 1, \dots, N$. Thus $\mathcal{L}^n(B_1(\xi) \cap \{|u(\xi) - a_j| > \gamma/2\}) \geq \mu_0$, $j = 1, \dots, N$ for some $\mu_0 > 0$ and so by Theorem A

$$(5.3) \quad \mathcal{L}^n(B_R(\xi) \cap \{|u(\xi) - a_j| > \frac{\gamma}{2}\}) \geq CR^n, \quad R \geq 1, \quad j = 1, \dots, N.$$

This and the same argument as in the proof of Theorem 2.1 imply

$$\sum_{i \neq j} \mathcal{L}^n(B_R(\xi) \cap \{|u(\xi) - a_i| < \frac{\gamma}{2}\}) \geq CR^n, \quad R \geq 1, \quad j = 1, \dots, N.$$

It follows that, for each $R \geq 1$, at least for two distinct $a_-, a_+ \in \{a_1, \dots, a_N\}$ we have

$$(5.4) \quad \mathcal{L}^n(B_R(\xi) \cap \{|u - a_{\pm}| < \frac{\gamma}{2}\}) \geq CR^n.$$

Define $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ by setting $\varphi(x) = d(a_-, u(x))$ with $d(z_1, z_2)$ ¹ given by.

$$d(z_1, z_2) = \inf \left\{ \int_0^1 \sqrt{2W(\zeta(s))} |\zeta'(s)| ds, \zeta \in C^1([0, 1]; \mathbb{R}^m), \zeta(0) = z_1, \zeta(1) = z_2 \right\}.$$

By (2.5) we have $d(a_\pm, z) \leq C|z - a_\pm|$ with $C = \max_i \max_{|z - a_i| \leq M} \sqrt{2W(z)}$. It follows that, provided $\gamma \in (0, d(a_-, a_+)/C)$,

$$(5.5) \quad \begin{aligned} \{|u - a_-| < \frac{\gamma}{2}\} &\subset \{d(a_-, u) < t\}, \\ \{|u - a_+| < \frac{\gamma}{2}\} &\subset \{d(a_-, u) > t\}, \end{aligned} \quad \text{for } t \in (C\frac{\gamma}{2}, d(a_-, a_+) - C\frac{\gamma}{2}).$$

This and the relative isoperimetric inequality (see [16] pag. 190)

$$\min\{\mathcal{L}^n(B_R(\xi) \cap \{\varphi < t\}), \mathcal{L}^n(B_R(\xi) \setminus \{\varphi < t\})\}^{\frac{n-1}{n}} \leq C\mathcal{H}^{n-1}(B_R(\xi) \cap \varphi^{-1}(t))$$

imply via (5.4) the estimate

$$(5.6) \quad CR^{n-1} \leq \mathcal{H}^{n-1}(B_R(\xi) \cap \varphi^{-1}(t)) \quad \text{for } t \in (\alpha, \beta)$$

where $\alpha = C\frac{\gamma}{2}$ and $\beta = d(a_-, a_+) - C\frac{\gamma}{2}$. From Proposition 2.1 in [10] and (2.5) we have that φ is lipschitz and

$$(5.7) \quad \int_A |D\varphi| dx \leq \int_A \sqrt{2W(u)} |Du| dx$$

for all bounded smooth open subsets $A \subset \mathbb{R}^n$. Therefore by the coarea formula, the estimate (5.6), and Young's inequality we obtain

$$\begin{aligned} CR^{n-1} &\leq \int_\alpha^\beta \mathcal{H}^{n-1}(B_R(\xi) \cap \varphi^{-1}(t)) dt = \int_{B_R(\xi) \cap \{\alpha < \varphi(x) < \beta\}} |D\varphi| dx \\ &\leq \int_{B_R(\xi)} \sqrt{2W(u)} |Du| dx \leq J_{B_R(\xi)}(u) \end{aligned}$$

that concludes the proof. □

We give another proof of Proposition 5.1 via linking with the sharp interface problem in [10].

Proof. (Blow-down) Let $\xi \in \mathbb{R}^n$ as before and set $x - \xi = \frac{y}{\epsilon}$, $u_\epsilon(y) = u(\xi + \frac{y}{\epsilon})$. Then (2.35) implies

$$(5.8) \quad \int_{|y| < r} \left(\frac{\epsilon}{2} |\nabla u_\epsilon|^2 + \frac{1}{\epsilon} W(u_\epsilon) \right) dy \leq Cr^{n-1}, \quad \text{for } \epsilon \in (0, 1),$$

where $r = \epsilon R$ is fixed once and for all. From (2.5) and (5.8) it follows (see pages 73, 82 in [10]) that $\|u_\epsilon\|_{BV(B_r(0); \mathbb{R}^m)} < C$ and so along a subsequence

$$u_{\epsilon_k} \xrightarrow{L^1} u_0 \quad \text{in } B_r(0) \quad \text{and} \quad u_0(y) \in \{a_1, \dots, a_N\} \quad \text{a.e.}$$

¹ $d(z_1, z_2)$ is the geodesic distance ([10] pag. 71)

Moreover $A_j = \{u_0(y) = a_j\}$, $j = 1, \dots, N$ are sets of finite perimeter in $B_r(0)$. From (5.3) we have **

$$\mathcal{L}^n(B_r(0) \cap \{|u_{\epsilon_k} - a_j| > \frac{\gamma}{2}\}) \geq C(\epsilon_k R)^n = Cr^n, \quad \text{for } j = 1, \dots, N.$$

and by passing to the limit for $k \rightarrow \infty$ we obtain

$$\mathcal{L}^n(B_r(0) \cap \{|u_0 - a_j| > \frac{\gamma}{2}\}) \geq Cr^n, \quad \text{for } j = 1, \dots, N.$$

Hence

$$\sum_{i \neq j} \mathcal{L}^n(B_r(0) \cap A_i) \geq Cr^n, \quad \text{for } j = 1, \dots, N.$$

From this it follows that at least for two distinct values $a_h \neq a_l$ the sets A_h, A_l have full measure:

$$\mathcal{L}^n(B_r(0) \cap A_h) \geq \frac{C}{N-1}r^n, \quad \mathcal{L}^n(B_r(0) \cap A_l) \geq \frac{C}{N-1}r^n.$$

Then the relative isoperimetric inequality implies

$$(5.9) \quad \mathcal{H}^{n-1}(\partial A_h) \geq Cr^{n-1}, \quad \mathcal{H}^{n-1}(\partial A_l) \geq Cr^{n-1}$$

where ∂A_j is the relative boundary of A_j in $B_r(0)$ and $C > 0$ a constant. Finally by lower semicontinuity (see pag.76 in [10]) and (5.9) we have

$$(5.10) \quad \liminf_{k \rightarrow \infty} \int_{|y| < r} \left(\frac{\epsilon_k}{2} |\nabla u_{\epsilon_k}|^2 + \frac{1}{\epsilon_k} W(u_{\epsilon_k}) \right) dy \geq \sum_{i,j=1, i \neq j}^N d(a_i, a_j) \mathcal{H}^{n-1}(\partial A_i \cap \partial A_j) \\ \geq C \mathcal{H}^{n-1}(\partial A_h) \geq Cr^{n-1}.$$

Since the right hand side of (5.10) is independent of the particular subsequence $\{\epsilon_k\}$ considered, we conclude that there is $\epsilon_0 > 0$ such that

$$\int_{|y| < r} \left(\frac{\epsilon}{2} |\nabla u_\epsilon|^2 + \frac{1}{\epsilon} W(u_\epsilon) \right) dy \geq C \left(\frac{r}{2} \right)^{n-1}, \quad \text{for } \epsilon \in (0, \epsilon_0)$$

and in the original variables

$$(5.11) \quad \int_{B_R(\xi)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \geq CR^{n-1}, \quad \text{for } R \geq R_0.$$

To conclude the proof we show that given $x_0 \in \mathbb{R}^n$ there is $R(x_0)$ such that

$$J_{B_R(x_0)}(u) \geq \frac{C}{2} R^{n-1}, \quad \text{for } R \geq R(x_0)$$

where $C > 0$ is the constant in (5.11). Indeed from (5.11), for $R \geq R_0 + |x_0 - \xi|$, we have, with $d = |x_0 - \xi|$,

$$J_{B_R(x_0)}(u) \geq J_{B_{R-d}(\xi)}(u) \geq C(R-d)^{n-1} \geq \frac{C}{2} R^{n-1} \quad \text{for } R \geq R(x_0) = \frac{2^{\frac{1}{n-1}}}{2^{\frac{1}{n-1}} - 1} d.$$

This completes the proof. \square

PART II

6 Theorem B

6.1 Hypotheses and statement

In this subsection we consider

$$(6.1) \quad \Delta u - W_u(u) = 0, \quad u : \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

in the class of *symmetric* solutions

$$u(\hat{x}) = \hat{u}(x)$$

where for $z \in \mathbb{R}^d$, $d \geq 1$ we denote by \hat{z} the symmetric of z in the plane $\{z_1 = 0\}$ that is

$$\hat{z} = (-z_1, z_2, \dots, z_d).$$

We assume that $W : \mathbb{R}^m \rightarrow \mathbb{R}$ is a C^3 potential that satisfies

(ha) W is symmetric: $W(\hat{u}) = W(u)$, for $u \in \mathbb{R}^m$ and

$$(6.2) \quad 0 = W(a_+) < W(u), \quad \text{for } u \in \overline{\mathbb{R}_+^m} \setminus \{a_+\}$$

for a unique $a_+ \in \mathbb{R}_+^m = \{u \in \mathbb{R}^m : u_1 > 0\}$.

$$(6.3) \quad W_{uu}(a_+)\nu \cdot \nu \geq C_0 > 0, \quad \text{for } |\nu| = 1,$$

where $W_{uu}(a_+)$ is the Hessian matrix of W at a_+ .

(hb) There exists $e : \mathbb{R} \rightarrow \mathbb{R}^m$ (*connection*) satisfying

$$(6.4) \quad \begin{cases} e_{ss} = W_u(e), & s \in \mathbb{R} \\ e(-s) = \hat{e}(s), & s \in \mathbb{R}, \\ \lim_{s \rightarrow \pm\infty} e(s) = a_{\pm}, \end{cases}$$

which moreover is a *global minimizer* of the *Action* functional

$$A(v) = \int_{\mathbb{R}} \left(\frac{1}{2} |v_s|^2 + W(v) \right) ds$$

in the class of $v \in W_{\text{loc}}^{1,2}(\mathbb{R}; \mathbb{R}^m) \cap L^\infty(\mathbb{R}; \mathbb{R}^m)$ which are symmetric and satisfy $\lim_{s \rightarrow \pm\infty} v(s) = a_{\pm}$.

The connection e is *hyperbolic* in the class of *symmetric* perturbations in the sense that the operator T defined by

$$(6.5) \quad D(T) = W_S^{2,2}(\mathbb{R}, \mathbb{R}^m), \quad Tv = -v'' + W_{uu}(e)v,$$

where $W_S^{2,2}(\mathbb{R}, \mathbb{R}^m) \subset W^{2,2}(\mathbb{R}, \mathbb{R}^m)$ is the subspace of symmetric maps, satisfies

$$(6.6) \quad \langle Tv, v \rangle \geq \eta \|v\|^2, \quad v \in W_S^{1,2}(\mathbb{R}, \mathbb{R}^m).$$

for some $\eta > 0$. Here $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R}; \mathbb{R}^m)$ and $\| \cdot \|$ the associated norm and W_{uu} is the Hessian matrix of W .

(hc) $u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$, is *minimal* in the class of symmetric maps in the sense that

$$J_\Omega(u) \leq J_\Omega(u + v), \text{ for each symmetric } v \in W_0^{1,2}(\Omega; \mathbb{R}^m)$$

and for every open symmetric bounded lipschitz set $\Omega \subset \mathbb{R}^n$. Moreover u satisfies the estimates

$$(6.7) \quad |u - a| + |\nabla u| \leq K e^{-kx_1}, \text{ on } \overline{\mathbb{R}_+^n}$$

for some $k, K > 0$.

Since we have

$$(6.8) \quad |e - a| + |u_{x_1}| \leq K e^{-kx_1}, \text{ on } x_1 \geq 0,$$

it follows, via (6.7), that

$$(6.9) \quad \|u(\cdot, x_2, \dots, x_n) - e(\cdot)\|_{W^{1,2}(\mathbb{R}; \mathbb{R}^m)} \leq M_1, \text{ for } (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$$

for some constant $M_1 > 0$.

We denote $E^{\text{xp}} \subset W_{S, \text{loc}}^{1,2}(\mathbb{R}; \mathbb{R}^m)$ the exponential class of symmetric maps which, as e , satisfy (6.8) with $k, K > 0$ fixed constants.

Notes

- (i) Under hypotheses (ha) by Theorems 3.6, 3.7 in [7] there is a connection e symmetric and global minimizer of A .
- (ii) In the proof of Theorem B we utilize minimality only in symmetric cylinders.

Notation

As before by \cdot we denote the Euclidean inner product in \mathbb{R}^d $d \geq 2$. We write the typical $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ in the form $x = (s, y)$ with $s = x_1 \in \mathbb{R}$ and $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. For $r > 0$ and $y_0 \in \mathbb{R}^{n-1}$ we set $\mathcal{B}_r(y_0) = \{y \in \mathbb{R}^{n-1} : |y - y_0| < r\}$. By $C_r(y_0) \subset \mathbb{R}^n$ we denote the cylinder $\mathbb{R} \times \mathcal{B}_r(y_0)$.

Theorem B. *Under hypothesis (ha), (hb) and (hc) above, there exists $\lambda^* > 0$ small, independent of u , such that for any $\mu_0 > 0$ and any $0 < \lambda < \lambda^*$ the condition*

$$(6.10) \quad \mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e(\cdot)\| \geq \lambda\}) \geq \mu_0$$

implies the estimate

$$(6.11) \quad \mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e(\cdot)\| \geq \lambda\}) \geq C R^{n-1}, \text{ for } R \geq 1$$

where $C = C(\mu_0, \lambda, K)$, is independent of y_0 and independent of u .

Theorem B has the following important consequence

Theorem 6.1. *Assume that W satisfies (ha) and that $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is minimal in the sense of (hc). Assume that there are exactly two global minimizers $e_+ \neq e_-$ of the action A in the symmetric class with the properties of e in (hb) above. Then the condition*

$$(6.12) \quad \mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \leq \theta\}) \geq \mu_0 > 0$$

$\theta \in (0, \|e_+ - e_-\|)$, arbitrary otherwise, implies the estimate

$$(6.13) \quad \mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \leq \theta\}) \geq C R^{n-1}, \text{ for } R \geq 1,$$

where $C = C(\mu_0, \lambda, K)$, is independent of y_0 and independent of u . An analogous statement applies to e_- .

6.2 The Proof of Theorem B

1. The Polar Form and the Effective Potential

We will utilize the *polar form with respect to e* of a vector map $u \in W_{\text{loc}}^{1,2}(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$. We write

$$u(s, y) = e(s) + q^u(y)\nu^u(s, y), \quad (s, y) \in \mathbb{R}^n$$

where

$$q^u(y) = \|u(\cdot, y) - e(\cdot)\|$$

and

$$(6.14) \quad \nu^u(\cdot, y) = \begin{cases} \frac{u(\cdot, y) - e(\cdot)}{\|u(\cdot, y) - e(\cdot)\|} & \text{if } q^u(y) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$(6.15) \quad \frac{\partial u}{\partial y_i} = \frac{\partial q^u}{\partial y_i} \nu^u + q^u \frac{\partial \nu^u}{\partial y_i}$$

and therefore observing that

$$(6.16) \quad \|\nu^u(\cdot, y)\| = 1, \quad \langle \nu^u(\cdot, y), \frac{\partial \nu^u}{\partial y_i}(\cdot, y) \rangle = 0, \quad i = 1, \dots, n-1,$$

we obtain the following *polar representation* of the energy of u

$$(6.17) \quad \begin{aligned} & \int_{C_r(y_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \\ &= \int_{\mathcal{B}_r(y_0)} \left(\frac{1}{2} (|\nabla q^u|^2 + (q^u)^2 \sum_{i=1}^{n-1} \left\| \frac{\partial \nu^u}{\partial y_i} \right\|^2) + \mathcal{W}(u) + A(e) \right) dy \end{aligned}$$

where $\mathcal{W} : e + W^{1,2}(\mathbb{R}; \mathbb{R}^m) \rightarrow \mathbb{R}$ the *Effective Potential* is defined by

$$(6.18) \quad \begin{aligned} \mathcal{W}(v) &= A(v) - A(e) \\ &= \int_{\mathbb{R}} \left(\frac{1}{2} (\|v_s\|^2 - \|e_s\|^2) + W(v) - W(e) \right) ds, \quad \text{for } v - e \in W_S^{1,2}(\mathbb{R}; \mathbb{R}^m). \end{aligned}$$

As it is standard in variational arguments, adding a constant to the integrand in (6.17) does not affect what follows. Therefore we disregard the constant $A(e)$ in (6.17) and define the modified energy $J_{C_r(y_0)}(u)$ by setting

$$(6.19) \quad J_{C_r(y_0)}(u) = \int_{\mathcal{B}_r(y_0)} \left(\frac{1}{2} (|\nabla q^u|^2 + (q^u)^2 \sum_{i=1}^{n-1} \left\| \frac{\partial \nu^u}{\partial y_i} \right\|^2) + \mathcal{W}(u) \right) dy$$

where we have slightly abused the notation in (2.4). Note that

$$(6.20) \quad J_{C_r(y_0)}(e) = 0.$$

Lemma 6.2. *We have*

(i) $\mathcal{W} \geq 0$.

(ii) *Let $\mathbb{S} = W^{1,2}(\mathbb{R}; \mathbb{R}^m) \cap \{\|v\| = 1\}$. Assume that $v(s) = e(s) + q\nu$, $q \in \mathbb{R}$, $\nu \in \mathbb{S}$ satisfies (6.9). Then there are constants $c_0 > 0$ and $\bar{q} > 0$ such that*

$$(6.21) \quad D_{qq}\mathcal{W}(e + q\nu) \geq c_0, \quad \text{for } q \in [0, \bar{q}], \nu \in \mathbb{S}.$$

Proof. (i) follows from (hb). To prove (ii) we begin by differentiating twice $\mathcal{W}(e + q\nu)$ with respect to q . We obtain

$$(6.22) \quad \begin{aligned} D_{qq}\mathcal{W}(e + q\nu) &= \|\nu_s\|^2 + \int_{\mathbb{R}} W_{uu}(e + q\nu)\nu \cdot \nu ds \\ &= D_{qq}\mathcal{W}(e + q\nu)|_{q=0} + \int_{\mathbb{R}} (W_{uu}(e + q\nu) - W_{uu}(e))\nu \cdot \nu ds. \end{aligned}$$

From the interpolation inequality:

$$(6.23) \quad \begin{aligned} \|f\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} &\leq \sqrt{2}\|f\|^{\frac{1}{2}}\|f_s\|^{\frac{1}{2}}, \quad f \in W^{1,2}(\mathbb{R}; \mathbb{R}^m), \\ &\leq \sqrt{2}\|f\|_{W^{1,2}(\mathbb{R}; \mathbb{R}^m)}, \end{aligned}$$

applied to $q\nu$ we obtain via the second inequality

$$(6.24) \quad \|q\nu\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} \leq \sqrt{2}M_1,$$

with M_1 the constant in (6.9), and via the first

$$(6.25) \quad \|\nu\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} \leq \sqrt{2}M_1^{\frac{1}{2}}q^{-\frac{1}{2}},$$

since $\|q\nu\| = q$ and $\|q\nu_s\| \leq M_1$. Therefore we have

$$(6.26) \quad |W_{u_i u_j}(e(s) + q\nu(s)) - W_{u_i u_j}(e(s))| \leq \sqrt{2}M_1^{\frac{1}{2}}\overline{W}''' q^{\frac{1}{2}},$$

where \overline{W}''' is defined by

$$(6.27) \quad \overline{W}''' := \max_{\substack{1 \leq i, j, k \leq m \\ s \in \mathbb{R}, |\tau| \leq 1}} W_{u_i u_j u_k}(e(s) + \tau\sqrt{2}M_1).$$

From (6.26) we get

$$(6.28) \quad \left| \int_{\mathbb{R}} (W_{uu}(e + q\nu) - W_{uu}(e))\nu \cdot \nu ds \right| \leq C_1 q^{\frac{1}{2}} \langle \nu, \nu \rangle = C_1 q^{\frac{1}{2}},$$

where $C_1 > 0$ is a constant that depends on M_1 . We now observe that

$$(6.29) \quad D_{qq}\mathcal{W}(e + q\nu)|_{q=0} = \langle T\nu, \nu \rangle \geq \eta\|\nu\|^2 = \eta$$

where we have also utilized (6.6). Thus (6.29) and (6.28) in (6.22) yield

$$(6.30) \quad D_{qq}\mathcal{W}(e + q\nu) \geq c_0 := \frac{\eta}{2}, \quad \text{for } q \in [0, \bar{q}],$$

where $\bar{q} = \frac{1}{4} \frac{\eta^2}{C_1^2}$. This concludes the proof of the lemma. \square

In the following lemma we show that in the definition of minimality in (hc) we can extend the class of sets to include unbounded cylinders aligned to the x_1 axis.

Lemma 6.3. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be minimal as in (hc) above. Given a bounded open set $O \subset \mathbb{R}^{n-1}$, we have*

$$(6.31) \quad J_{\mathbb{R} \times O}(u) = \min_{v \in u + W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)} J_{\mathbb{R} \times O}(v),$$

where $W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ is the closure in $W_S^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ of the smooth maps v that satisfy $v = 0$ on $\mathbb{R} \times \partial O$.

Proof. Assume there are $\eta > 0$ and $v \in u + W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ such that

$$(6.32) \quad J_{\mathbb{R} \times O}(u) - J_{\mathbb{R} \times O}(v) \geq \eta.$$

For each $l > 0$ define $\tilde{v} \in W_{0S}^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ by

$$\tilde{v} = \begin{cases} v, & \text{for } s \in [0, l], y \in O, \\ (1 + l - s)v + (s - l)u, & s \in [l, l + 1], y \in O, \\ u, & \text{for } s \in [l + 1, +\infty), y \in O. \end{cases}$$

The minimality of u implies

$$(6.33) \quad 0 \geq J_{[-l-1, l+1] \times O}(u) - J_{[-l-1, l+1] \times O}(\tilde{v}) = J_{[-l-1, l+1] \times O}(u) - J_{[-l, l] \times O}(v) + O(e^{-kl}),$$

where we have also used the fact that both u and v belong to $W_S^{1,2}(\mathbb{R} \times O; \mathbb{R}^m)$ and satisfy (6.7). Taking the limit for $l \rightarrow +\infty$ in (6.33) yields

$$0 \geq J_{\mathbb{R} \times O}(u) - J_{\mathbb{R} \times O}(v),$$

in contradiction with (6.32). \square

For $q^h \in W^{1,2}(C_R; \mathbb{R}^m) \cap L^\infty(C_R; \mathbb{R}^m)$, $q^h \geq 0$, let the map σ defined via $\sigma = e + q^\sigma \nu^u$, $q^\sigma = \min\{q^h, q^u\}$. We have $\sigma \in W^{1,2}(C_R; \mathbb{R}^m) \cap L^\infty(C_R; \mathbb{R}^m)$ [9]. The minimality of u and the polar form (6.19) of the energy imply the inequality

$$(6.34) \quad \begin{aligned} & \frac{1}{2} \int_{\mathcal{B}_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dy \\ &= J_{C_R}(u) - J_{C_R}(\sigma) + \frac{1}{2} \int_{\mathcal{B}_R} \left((q^\sigma)^2 - (q^u)^2 \right) \sum_i^{n-1} \left\| \frac{\partial \nu^u}{\partial y_i} \right\|^2 dy + \int_{\mathcal{B}_R} (\mathcal{W}(\sigma) - \mathcal{W}(u)) dy \\ &\leq \int_{\mathcal{B}_R} (\mathcal{W}(\sigma) - \mathcal{W}(u)) dy. \end{aligned}$$

Indeed minimality and Lemma 6.3 imply $J_{C_R}(u) - J_{C_R}(\sigma) \leq 0$ and the second term is also nonpositive by $0 \leq q^\sigma \leq q^u$.

2. An Upper Bound for the Energy

Next we establish the analogous of Lemma 2.2 that is

Lemma 6.4. *Assume that W satisfies (ha) and assume that u is minimal as defined in (hc) and e a global minimizer of the Action as in (hb) above (hyperbolicity is not required). Then there is a constant $C > 0$ depending on K , independent of u and independent of y_0 such that*

$$(6.35) \quad \begin{aligned} 0 &\leq \int_{C_r(y_0)} \left(\frac{1}{2} |\nabla u|^2 + W(u) - \left(\frac{1}{2} |e_s|^2 + W(e) \right) \right) dx \\ &= J_{C_R(y_0)}(u) \leq CR^{n-2}, \quad \text{for } R > 0. \end{aligned}$$

Proof. Let

$$(6.36) \quad v(\cdot, y) = \begin{cases} e(\cdot), & \text{for } y \in \mathcal{B}_{R-1}(y_0), \\ e(\cdot) + (|y - y_0| - R + 1)q^u(y)\nu^u(\cdot, y), & \text{for } y \in \mathcal{B}_R(y_0) \setminus \mathcal{B}_{R-1}(y_0). \end{cases}$$

From Lemma 6.3 we have

$$J_{C_R(y_0)}(u) \leq J_{C_R(y_0)}(v) = J_{\mathbb{R} \times (\mathcal{B}_R(y_0) \setminus \mathcal{B}_{R-1}(y_0))}(v),$$

and via (6.7)

$$J_{\mathbb{R} \times (\mathcal{B}_R(y_0) \setminus \mathcal{B}_{R-1}(y_0))}(v) \leq C\mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \setminus \mathcal{B}_{R-1}(y_0)) \leq CR^{n-2}.$$

The proof of the lemma is complete. \square

3. The Isoperimetric Inequality for Minimizers

As in the proof of the case $\alpha = 2$ in Theorem A we let $\varphi : \mathcal{B}_R \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ be the solution of the problem

$$(6.37) \quad \begin{cases} \Delta \varphi = c_1 \varphi, & \text{on } \mathcal{B}_R, \\ \varphi = 1, & \text{on } \partial \mathcal{B}_R, \end{cases}$$

where $c_1 < c_0$ will be chosen later and c_0 is the constant in Lemma 6.2. We set

$$q_M = \sup_{y \in \mathbb{R}^{n-1}} \|u(\cdot, y)\|$$

and define

$$(6.38) \quad \begin{aligned} h &= e + q^h \nu^u, & q^h &= \varphi q_M, \text{ and as before} \\ \sigma &= e + q^\sigma \nu^u, & q^\sigma &= \min\{q^u, q^h\}, \\ \beta &= \min\{q^u - q^\sigma, \lambda\}, \end{aligned}$$

where $\lambda \in (0, \bar{q})$ with \bar{q} as in Lemma 6.2. We also recall the exponential estimate

$$(6.39) \quad \varphi(R - r) \leq e^{-c_2(R-r)}, \quad \text{for } r \in [0, R], \quad R \geq 1,$$

for some $c_2 > 0$.

We remark that the definition of σ in (6.38) implies

$$q^\sigma = q^u, \quad \text{on } \partial \mathcal{B}_R$$

and that $\sigma \in W^{1,2}(C_R; \mathbb{R}^m) \cap L^\infty(C_R; \mathbb{R}^m)$ (see [9]).

Proceeding as in the proof of Theorem A by applying the inequality in [16] on $\mathcal{B}_R \subset \mathbb{R}^{n-1}$ to β^2 yields

$$\begin{aligned}
(6.40) \quad & \left(\int_{\mathcal{B}_R} \beta^{\frac{2(n-1)}{n-2}} dy \right)^{\frac{n-2}{n-1}} = \left(\int_{\mathcal{B}_R} (\beta^2)^{\frac{n-1}{n-2}} dy \right)^{\frac{n-2}{n-1}} \\
& \leq C \int_{\mathcal{B}_R} |\nabla(\beta^2)| dy \quad (\beta = 0, \text{ on } \partial B_R) \\
& \leq 2C \int_{\mathcal{B}_R \cap \{q^u - q^\sigma \leq \lambda\}} |\nabla \beta| |\beta| dy \\
& \leq CA \int_{\mathcal{B}_R} |\nabla(q^u - q^\sigma)|^2 dy + \frac{C}{A} \int_{\mathcal{B}_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dy \\
& = CA \left(\int_{\mathcal{B}_R} (|\nabla q^u|^2 - |\nabla q^\sigma|^2) dy - 2 \int_{\mathcal{B}_R} \nabla q^\sigma \cdot \nabla(q^u - q^\sigma) dy \right) \\
& \quad + \frac{C}{A} \int_{\mathcal{B}_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dy
\end{aligned}$$

where we have utilized $\nabla \beta = 0$ a.e. on $q^u - q^\sigma > \lambda$ and Young's inequality. Thus via (6.34) we derive

$$\begin{aligned}
(6.41) \quad & \left(\int_{\mathcal{B}_R} \beta^{\frac{2n-1}{n-2}} dy \right)^{\frac{n-2}{n-1}} \leq \\
& \leq 2CA \left(\int_{\mathcal{B}_R} (\mathcal{W}(\sigma) - \mathcal{W}(u)) dy - \int_{\mathcal{B}_R} \nabla q^\sigma \cdot \nabla(q^u - q^\sigma) dy \right) \\
& \quad + \frac{C}{A} \int_{\mathcal{B}_R \cap \{q^u - q^\sigma \leq \lambda\}} (q^u - q^\sigma)^2 dy
\end{aligned}$$

4. Conclusion

The inequality (6.41), aside from the fact that n is replaced by $n-1$, \mathcal{B}_R is the ball of radius R in \mathbb{R}^{n-1} and W is replaced by \mathcal{W} , coincides with (2.19). Moreover, by Lemma 6.2, \mathcal{W} has the properties of W in (HA), $\alpha = 2$ and Lemma 6.4 is the counterpart of Lemma 2.2. The only difference is that the inequality

$$W(h) - W(u) \leq W(h)$$

now is replaced by

$$\mathcal{W}(h) - \mathcal{W}(u) \leq \mathcal{W}(h).$$

Thus the arguments developed in the proof of Theorem A for the case $\alpha = 2$ can be repeated verbatim to complete the proof of Theorem B.

6.3 The Proof of Theorem 6.1

1. First we note that under the hypotheses of Theorem 6.1 we can take $\lambda^* = \|e_+ - e_-\|$ in the statement of Theorem B. To argue this we let $\hat{\lambda} \in (0, \|e_+ - e_-\|)$ and assume that

$$\mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \geq \hat{\lambda}\}) \geq \mu_0 > 0.$$

Thus for $\lambda < \hat{\lambda}$, $\lambda > 0$ as in Theorem B and fixed

$$\mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \geq \lambda\}) \geq \mu_0 > 0.$$

Therefore

$$\mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \geq \lambda\}) \geq CR^{n-1}, \quad R \geq 1.$$

We will be done if we can show that

$$(6.42) \quad \mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \lambda \leq \|u(\cdot, y) - e_+(\cdot)\| \leq \hat{\lambda}\}) \leq CR^{n-2}.$$

For this purpose note that $\lambda \leq \|u(\cdot, y) - e_+(\cdot)\| \leq \hat{\lambda}$ implies

$$\|u(\cdot, y) - e_-(\cdot)\| \geq \|e_+ - e_-\| - \|u(\cdot, y) - e_+(\cdot)\| \geq \|e_+ - e_-\| - \hat{\lambda} = \tilde{\lambda} > 0.$$

Thus on $S_{\tilde{\lambda}}^{\hat{\lambda}} = \mathcal{B}_R(y_0) \cap \{y : \lambda \leq \|u(\cdot, y) - e_+(\cdot)\| \leq \hat{\lambda}\}$ we have the estimate

$$\mathcal{W}(u(\cdot, y)) \geq \bar{w}(\tilde{\lambda}) > 0$$

and so via (6.35)

$$\bar{w}(\tilde{\lambda}) \mathcal{L}^{n-1}(S_{\tilde{\lambda}}^{\hat{\lambda}}) \leq \int_{S_{\tilde{\lambda}}^{\hat{\lambda}}} \mathcal{W}(u(\cdot, y)) dy \leq CR^{n-2},$$

and so (6.42) is established.

2. Suppose now that

$$\mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \leq \theta\}) > \mu_0 > 0.$$

Since $\lambda = \|e_+ - e_-\| - \theta$ implies $\{y : \|u(\cdot, y) - e_-(\cdot)\| \leq \theta\} \subset \{y : \|u(\cdot, y) - e_-(\cdot)\| \geq \lambda\}$ it follows that

$$\mathcal{L}^{n-1}(\mathcal{B}_1(y_0) \cap \{y : \|u(\cdot, y) - e_-(\cdot)\| > \lambda\}) \geq \mu_0 > 0.$$

Hence by 1. above

$$\mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e_-(\cdot)\| > \lambda\}) \geq CR^{n-1}. \quad R \geq 1.$$

From this it easily follows that

$$\mathcal{L}^{n-1}(\mathcal{B}_R(y_0) \cap \{y : \|u(\cdot, y) - e_+(\cdot)\| \leq \theta\}) \geq CR^{n-1}.$$

The proof of Theorem 6.1 is complete.

6.4 On the Product Structure of Solutions

In this subsection we give alternative proofs of some of the results in [8].

Theorem 6.5. ([8], Theorem 1.2) *Assume that W satisfies (ha) and (hb) and assume that the connection e in (hb) is unique. Let $O \subset \mathbb{R}^{n-1}$, $O \neq \mathbb{R}^{n-1}$ be open with $\sup_{y_0 \in O} R_{y_0} = +\infty$ ($R_{y_0} = \sup_R \{\mathcal{B}_R(y_0) \subset O\}$) and assume that $u : \mathbb{R} \times O \rightarrow \mathbb{R}^m$ is minimal in the sense of (hc) (with \mathbb{R}^n replaced by $\mathbb{R} \times O$). Then there are constants $k_0, K_0 > 0$ such that*

$$|u(s, y) - e(s)| \leq K_0 e^{-k_0 d(y, \partial O)}.$$

Proof. It is sufficient to establish that, given a small number $\gamma > 0$, there is $d_\gamma > 0$ such that

$$(6.43) \quad d(y, \partial O) \geq d_\gamma \quad \Rightarrow \quad |u(s, y) - e(s)| < \gamma.$$

since then linear theory renders the result.

By Lemma 6.6 below there is a constant $C > 0$ such that

$$\|u(\cdot, y) - e(\cdot)\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} \leq C \|u(\cdot, y) - e(\cdot)\|^{\frac{2}{3}}.$$

Therefore $|u(s, y) - e(s)| \geq \gamma$ implies

$$(6.44) \quad \|u(\cdot, y) - e(\cdot)\| \geq \left(\frac{\gamma}{C}\right)^{\frac{3}{2}}.$$

From the assumed uniqueness and hyperbolicity of e it follows that, given $\lambda > 0$ small, it results

$$\|u(\cdot, y) - e(\cdot)\| \geq \lambda \quad \Rightarrow \quad \mathcal{W}(u(\cdot, y)) \geq \bar{w}(\lambda) > 0.$$

Therefore arguing as in the proof of Theorem 3.1 we deduce from Theorem B and Lemma 6.4 that there is $R(\lambda) > 0$ such that

$$(6.45) \quad \mathcal{B}_{R(\lambda)}(y_0) \subset O \quad \Rightarrow \quad \|u(\cdot, y_0) - e(\cdot)\| < \lambda$$

This and (6.44) imply that we can take $d_\gamma = R(\frac{\gamma}{C})^{\frac{3}{2}}$ in (6.43). The proof is complete. \square

We now establish

Lemma 6.6. (cfr. [8] Lemma 2.2) *Let $v \in E^{\text{xp}}$. Then*

$$(6.46) \quad \|v\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} \leq C \|v\|^{\frac{2}{3}},$$

where $C = C(k, K) > 0$ ((6.7)) is independent of v .

Proof.

$$\begin{aligned} |v(s)|^p &= \int_{-\infty}^s \frac{\partial}{\partial t} |v(t)|^p dt \leq p \int_{\mathbb{R}} |v(t)|^{p-1} |v_t(t)| dt \\ &\leq p \int_{\mathbb{R}} (|v(t)|^{p'(p-1)})^{\frac{1}{p'}} \left(\int_{\mathbb{R}} |v_t(t)|^{q'} dt \right)^{\frac{1}{q'}} \quad \left(\frac{1}{p'} + \frac{1}{q'} = 1 \right) \\ &\leq C \|v\|_{L^{p'(p-1)}(\mathbb{R}; \mathbb{R}^m)}^{p-1}. \end{aligned}$$

Hence

$$\|v\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)} \leq C \|v\|^{\frac{p-1}{p}}_{L^{p'(p-1)}(\mathbb{R}; \mathbb{R}^m)}.$$

Choosing first p' so that $p'(p-1) = 2$ and finally noting that $\max \frac{p-1}{p} = \frac{2}{3}$ we arrive at (6.46). The proof of the lemma is complete. \square

In [8] Theorem 6.5 was established by a different approach which also applies to a larger class of minimizers not necessarily defined on cylinders. We conclude with the following *Rigidity* result

Theorem 6.7. (see Theorem 1.3 in [8]) *Assume $u : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and otherwise the hypothesis of Theorem 6.5. Then*

$$u(x) = e(x_1), \text{ for } x \in \mathbb{R}^n.$$

Proof. The argument is essentially the same as in the proof of Theorem 3.2. For each $y_0 \in \mathbb{R}^{n-1}$ and for each $\lambda > 0$ we have trivially $\mathcal{B}_{R(\lambda)}(y_0) \subset \mathbb{R}^{n-1}$ and therefore, using also Lemma 6.6

$$C^{-\frac{3}{2}}(\|u(\cdot, y_0) - e(\cdot)\|_{L^\infty(\mathbb{R}; \mathbb{R}^m)})^{\frac{3}{2}} \leq \|u(\cdot, y_0) - e(\cdot)\| < \lambda, \text{ for each } y_0 \in \mathbb{R}^{n-1}, \lambda > 0.$$

The proof is concluded. □

References

- [1] S. Alama, L. Bronsard, and C. Gui. Stationary solutions in \mathbb{R}^2 for an Allen-Cahn system with multiple well potential, *Calc. Var. Part. Diff. Eqs.* **5** No. 4 (1997), pp. 359–390.
- [2] N. D. Alikakos. Some basic facts on the system $\Delta u - W_u(u) = 0$. *Proc. Amer. Math. Soc.* **139** No. 1 (2011), pp. 153–162.
- [3] N. D. Alikakos. On the structure of phase transition maps for three or more coexisting phases. In *Geometric partial differential equations*, M. Novaga and G. Orlandi eds. *Publications Scuola Normale Superiore*, CRM Series, Birkhauser, (2013).
- [4] N. D. Alikakos. A new proof for the existence of an equivariant entire solution connecting the minima of the potential for the system $\Delta u - W_u(u) = 0$. *Comm. Partial Diff. Eqs* **37** No. 12 (2012) pp. 2093–2115.
- [5] N. D. Alikakos and G. Fusco. Entire solutions to equivariant elliptic systems with variational structure. *Arch. Rat. Mech. Analysis* **202** No. 2 (2011), pp. 567–597.
- [6] N. D. Alikakos and G. Fusco. *in preparation*
- [7] N. D. Alikakos and G. Fusco. On the connection problem for potentials with several global minima. *Indiana Univ. Math. Journal* **57** (2008), pp. 1871–1906.
- [8] N. D. Alikakos and G. Fusco. Asymptotic and rigidity results for symmetric solutions of the elliptic system $\Delta u = W_u(u)$. arXiv:1402.5085.
- [9] N. D. Alikakos, G. Fusco and P. Smyrnelis. *Monograph* (in preparation).
- [10] S. Baldo. Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids. *Ann. Inst. Henri Poincaré* **7** No. 2 (1990), pp. 67–90.
- [11] F. Bethuel, H. Brezis and F. Helein. *Ginzburg-Landau Vortices*. Birkhauser (1994).
- [12] L. Bronsard and F. Reitich. On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation. *Arch. Rat. Mech. Analysis* **124** No. 4 (1993), pp. 355–379.

- [13] L. Bronsard, C. Gui, and M. Schatzman. A three-layered minimizer in \mathbb{R}^2 for a variational problem with a symmetric three-well potential. *Comm. Pure. Appl. Math.* **49** No. 7 (1996), pp. 677–715.
- [14] L. Caffarelli and A. Cordoba. Uniform convergence of a singular perturbation problem. *Comm. Pure Appl. Math.* **48** No. (1995), pp. 1–12.
- [15] L. Caffarelli and F. Lin. Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. *Journal Amer. Math. Society* **21** (2008), pp. 847–862.
- [16] L. C. Evans and R. F. Gariepy *Measure Theory and Fine Properties of Functions*. Studies in Advanced Mathematics. CRC Press New York (1992).
- [17] A. Farina. Two results on entire solutions of Ginzburg-Landau systems in higher dimensions. *J. Funct. Anal.* **214** No. 2 (2004), pp. 386–395.
- [18] A. Farina and E. Valdinoci. Geometry of quasiminimal phase transitions. *Calc. Var. Part. Diff. Eqs.* **33** No. (2008), pp. 1–35.
- [19] G. Fusco. Equivariant entire solutions to the elliptic system $\Delta u = W_u(u)$ for general G -invariant potentials. *Calc. Var. Part. Diff. Eqs.* February (2013), pp. 1–23.
- [20] G. Fusco. On some elementary properties of vector minimizers of the Allen-Cahn energy. *Comm. Pure Appl. Anal.* **13** No. 3 (2014), pp. 1045–1060.
- [21] E. Gonzalez, U. Massari and I. Tamanini. On the regularity of boundaries of sets minimizing perimeter with a volume constraint. *Indiana Univ. Math. Journal* **32** (1983), pp. 25–37.
- [22] C. Gui and M. Schatzman. Symmetric quadruple phase transitions. *Ind. Univ. Math. J.* **57** No. 2 (2008), pp. 781–83
- [23] O. Savin. Minimal Surfaces and Minimizers of the Ginzburg Landau energy. *Cont. Math. AMS* **526** (2010), pp. 43–58.
- [24] O. Savin and E. Valdinoci. Density estimates for a variational model driven by the Gagliardo norm. arXiv:1007.2114.
- [25] O. Savin and E. Valdinoci. Density estimates for a nonlocal variational model via the Sobolev inequality. arXiv:1103.6205.
- [26] Y. Sire and E. Valdinoci. Density estimates for phase transitions with a trace. arXiv: 1011.6617.
- [27] P. Smyrnelis. *personal communication*.
- [28] E. Valdinoci. Plane-like minimizers in periodic media: jet flows and Ginzburg-Landau-type functionals. *J. Reine Angew. Math.* **574** (2004), pp. 147–185.
- [29] B. White *Topics in GMT*. Notes by O.Chodosh. Stanford (2012).

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